

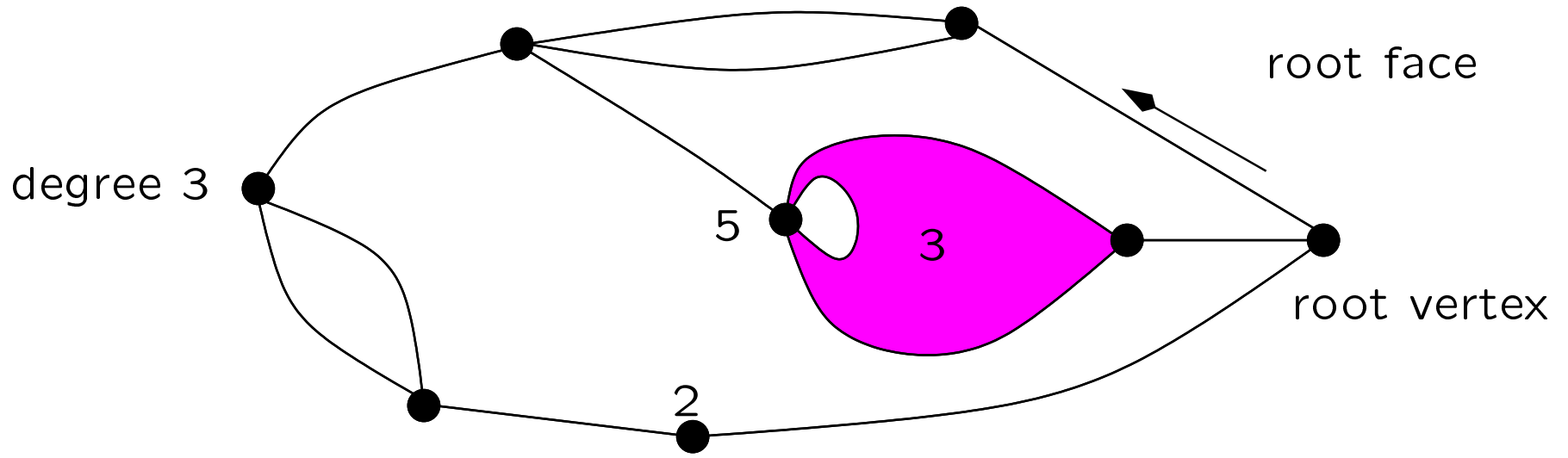
# The Potts model on planar maps

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## Rooted planar maps



- vertices  $V(M)$
- edges
- and faces

# Map enumeration

## In combinatorics, and in statistical physics

- The original question: how many maps with a given number of edges?
  - Tutte and his descendents (1960 → 2011)
  - Brézin-Itzykson-Parisi-Zuber and their descendents (1978 → 2011)

- Key object: the **generating function of planar maps**, counted by edges:

$$M(t) := \sum_M t^{e(M)} = 1 + 2t + O(t^2)$$

where  $e(M)$  is the number of edges of  $M$ .

- Refinement: maps counted by edges and vertex degrees:

$$M(t; x_1, x_2, x_3 \dots) := \sum_M t^{e(M)} x_1^{v_1(M)} x_2^{v_2(M)} x_3^{v_3(M)} \dots$$

where  $v_i(M)$  is the number of vertices of degree  $i$

## Maps equipped with an additional structure

In combinatorics, but mostly in statistical physics

### How many maps equipped with...

- a spanning tree?  
[Mullin 67]
- a spanning forest?  
[Bouttier et al., Sportiello et al.]
- a self-avoiding walk?  
[Duplantier-Kostov 88]
- a proper  $q$ -colouring?  
[Tutte 74, Bouttier et al. 02]

### What is the expected partition function of...

- the Ising model?  
[Boulatov, Kazakov, MBM, Schaeffer, Bouttier et al.]
- the hard-particle model?  
[MBM, Schaeffer, Jehanne, Bouttier et al. 02, 07]
- the Potts model?  
[Eynard-Bonnet 99, Baxter 01, MBM-Bernardi 09, Guionnet et al. 10]

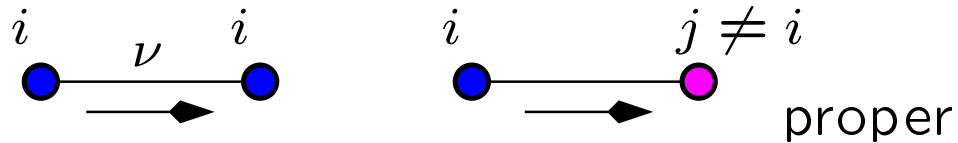
## The Potts model on planar maps

- The partition function of the  $q$ -state Potts model on a planar map  $M$ :

$$Z_M(q, \nu) = \sum_{c:V(M)\rightarrow\{1,2,\dots,q\}} \nu^{m(c)}$$

where  $m(c)$  is the number of monochromatic edges in the colouring  $c$ . In fact,  $Z_M(q, \nu)$  is a polynomial in  $q$  (and  $\nu$ ).

**Example:** When  $M$  has one edge and two vertices,  $Z_M(q, \nu) = q\nu + q(q - 1)$



## The Potts model on planar maps (cont'd)

- Generating function:

$$\begin{aligned}M(q, \nu, t) &= \sum_M Z_M(q, \nu) t^{e(M)} \\ &= \sum_{M, c} t^{e(M)} \nu^{m(c)} \\ &= q + (q\nu + q\nu + q(q-1))t + O(t^2)\end{aligned}$$

“The Potts generating function of planar maps”

⇒ Enumeration of  $q$ -coloured planar maps,  
counted by edges and monochromatic edges.

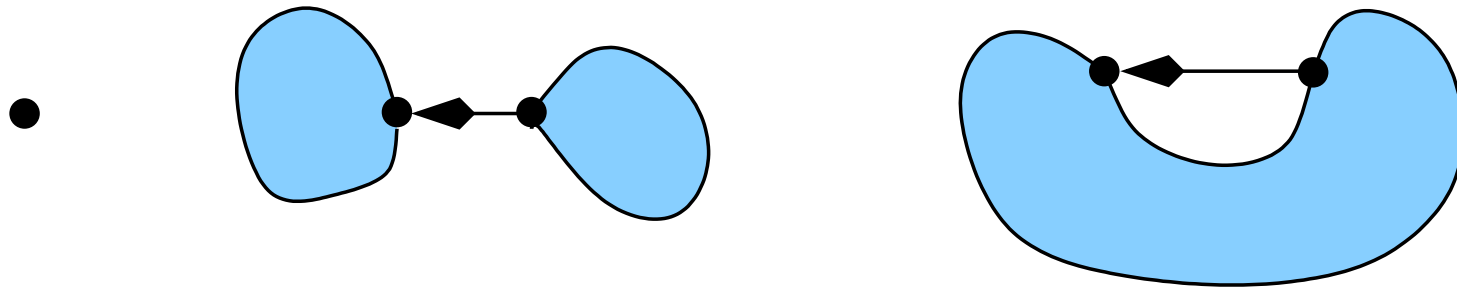
# I. Uncoloured planar maps: the recursive approach

## Recursive description of planar maps: deleting the root-edge

Let

$$F(t; x) \equiv F(x) = \sum_M t^{e(M)} x^{\text{df}(M)} = \sum_{d \geq 0} F_d(t) x^d$$

where  $e(M)$  is the number of edges and  $\text{df}(M)$  the degree of the outer face.



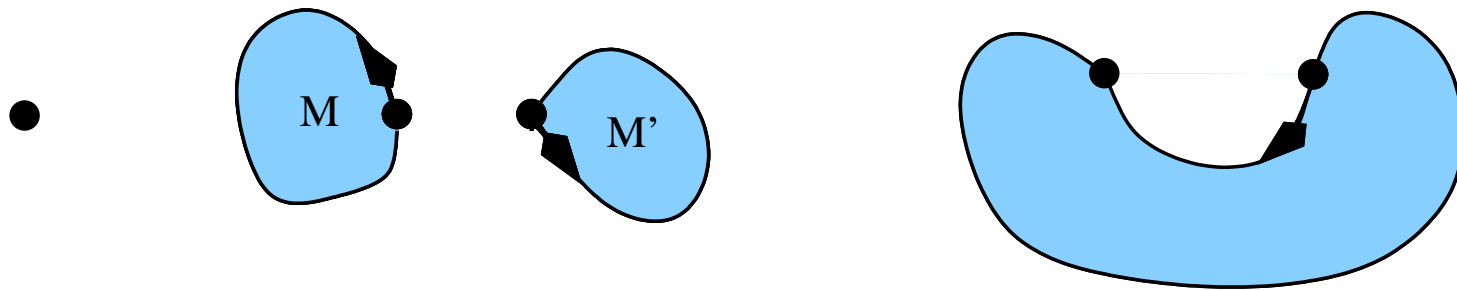


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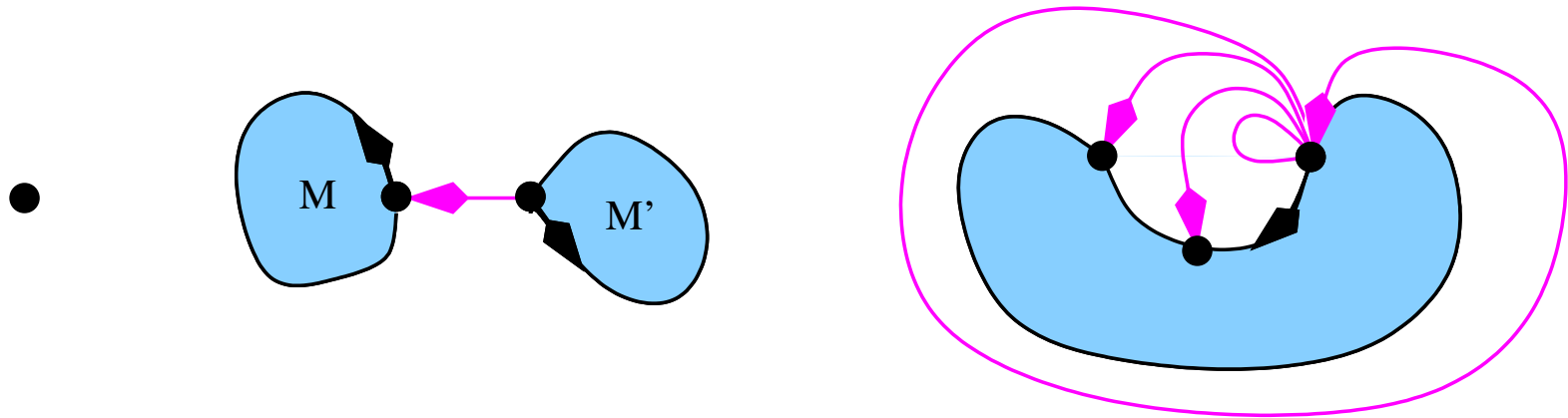


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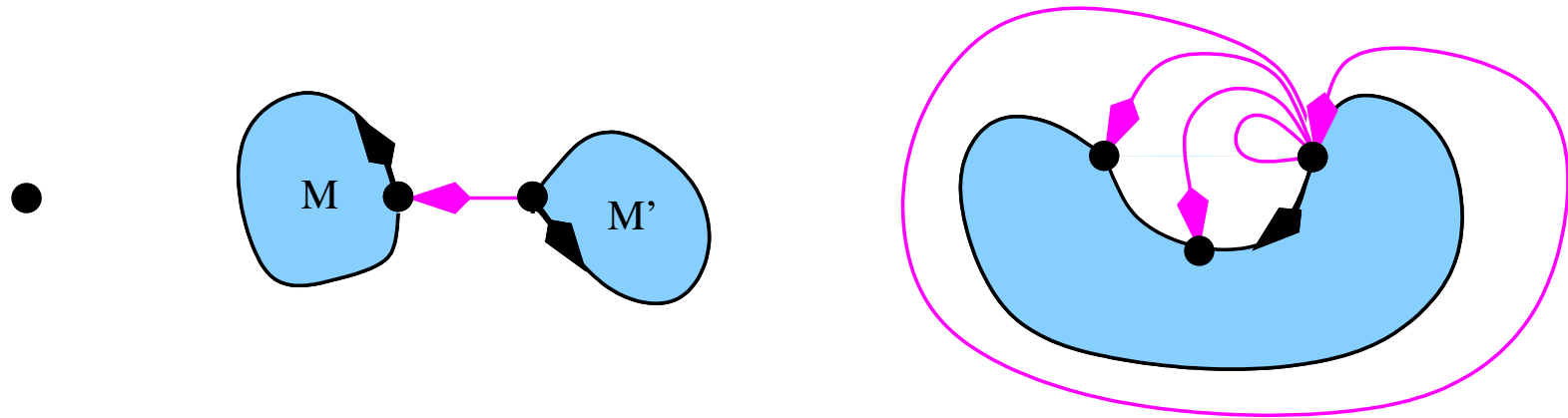


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$$\begin{aligned} F(x) &= 1 + tx^2 F(x)^2 + t \sum_{d \geq 0} F_d(t) (x^{d+1} + x^d + \dots + x) \\ &= 1 + tx^2 F(x)^2 + tx \frac{x F(x) - F(1)}{x - 1} \end{aligned}$$

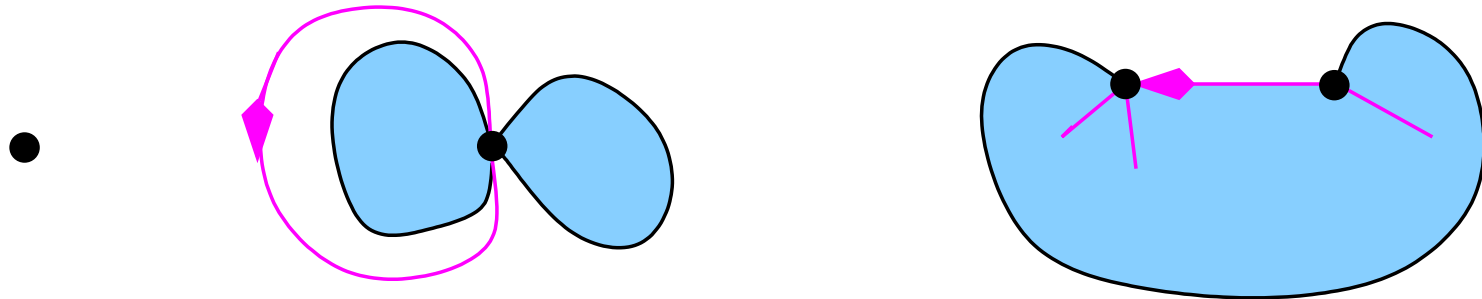
[Tutte 68] A quadratic equation with one **catalytic** variable,  $x$

## Recursive description of planar maps: contracting the root-edge

Let

$$F(t; y) \equiv F(y) = \sum_M t^{e(M)} y^{dv(M)} = \sum_{d \geq 0} F_d(t) y^d$$

where  $e(M)$  is the number of edges and  $dv(M)$  the degree of the root vertex.



$$\begin{aligned}
 F(y) &= 1 + ty^2 F(y)^2 + t \sum_{d \geq 0} F_d(t) (y^{d+1} + y^d + \dots + y) \\
 &= 1 + ty^2 F(y)^2 + ty \frac{yF(y) - F(1)}{y - 1}
 \end{aligned}$$

The same equation... (duality)

# Polynomial equations with one catalytic variable

Examples:

$$F(x) = 1 + tx^2F(x)^2 + tx \frac{x F(x) - F(1)}{x - 1}$$

$$F(x) = 1 + txF(x)^3 + tx(2F(x) + F(1)) \frac{F(x) - F(1)}{x - 1} + tx \frac{F(x) - F(1) - (x - 1)F'(1)}{(x - 1)^2}$$

Solution:

- Guess and check [Tutte 60's]
- Brown's quadratic method [Brown 65]
- A generalization to all polynomial equations with one catalytic variable [mbm-Jehanne 06]

## Polynomial equations with one catalytic variable

- **General framework:** A polynomial equation:

$$P(F(x), F_1, \dots, F_m, t, x) = 0 \quad (1)$$

where  $F(x) \equiv F(t; x)$  is a series in  $t$  with **polynomial** coefficients in  $x$ , and  $F_i \equiv F_i(t)$  does not depend on  $x$ .

- **Results**

1. The solution of every **proper** equation of this type is **algebraic**: There exists  $P$  such that  $P(t, x, F(t; x)) = 0$ .
2. A practical (but heavy) strategy allows to solve specific examples (that is, to derive from (1) an algebraic equation for  $F(x)$ , or  $F_1, \dots, F_m$ ).

[MBM-Jehanne 06]

## Example

$$F(t; x) \equiv F(x) = 1 + tx^2F(x)^2 + tx \frac{x F(x) - F(1)}{x - 1}$$

⇓

$$F(t; 1) = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2} = \sum_{n \geq 0} \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} t^n$$

with two lines of Maple.

Equivalently,

$$F(t; 1) = A(t) - tA(t)^3 \quad \text{where} \quad A(t) = 1 + 3tA(t)^2$$

⇒ Many map families have an algebraic generating function

## II. The Potts model on planar maps

### A recursive approach

$$M(q, \nu, t) = \sum_{M, c} t^{e(M)} \nu^{m(c)}$$

- Other approaches: [Eynard-Bonnet 99], [Guionnet et al. 10]



## Forget algebraicity!

One specialization of the Potts generating function  $M(q, \nu, t)$  counts planar maps equipped with a spanning tree, and...

**Theorem [Mullin 67]:** The number of planar maps of size  $n$  equipped with a spanning tree is

$$s_n = \frac{1}{(n+1)(n+2)} \binom{2n}{n} \binom{2n+2}{n+1} \sim \kappa 16^n n^{-3},$$

and this asymptotic behaviour prevents the series  $S(t) := \sum s_n t^n$  from being algebraic.

However, it satisfies a **linear differential equation**.

## Catalytic variables

The Potts generating function of planar maps, being transcendental, cannot be described with **one** catalytic variable

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WHY IS THAT SO?

- The recursive description of the Potts partition function

$$Z_G(q, \nu) = Z_{G \setminus e}(q, \nu) + (\nu - 1) Z_{G/e}(q, \nu)$$

calls for a recursive description of maps by contraction **and** deletion of edges.

- This is possible if one keeps track of the degree of the outer face, **and** the degree of the root-vertex.

## Equations with two catalytic variables

- Let

$$M(x, y) \equiv M(q, \nu, t; x, y) = \frac{1}{q} \sum_M Z_M(q, \nu) t^{e(M)} x^{dv(M)} y^{df(M)},$$

where  $dv(M)$  (resp.  $df(M)$ ) is the degree of the root-vertex (resp. root-face).

- The Potts generating function of planar maps satisfies:

$$\begin{aligned} M(x, y) = & 1 + xyt((\nu - 1)(y - 1) + qy) M(x, y)M(1, y) \\ & + xyt(x\nu - 1)M(x, y)M(x, 1) \\ & + xyt(\nu - 1) \frac{xM(x, y) - M(1, y)}{x - 1} + xyzt \frac{yM(x, y) - M(x, 1)}{y - 1}. \end{aligned}$$

[Tutte 68]

This equation has been sleeping for 40 years

## In the footsteps of W. Tutte

- For the GF  $T(q, t; x, y) \equiv T(x, y)$  of properly  $q$ -coloured triangulations:

$$T(x, y) = xy^2q(q-1) + \frac{xt}{yq}T(1, y)T(x, y) + xt \frac{T(x, y) - y^2T_2(x)}{y} - x^2yt \frac{T(x, y) - T(1, y)}{x-1}$$

where  $T_2(x)$  is the coefficient of  $y^2$  in  $T(x, y)$ .

- [Tutte 73] Chromatic sums for rooted planar triangulations: the cases  $\lambda = 1$  and  $\lambda = 2$
- [Tutte 73] Chromatic sums for rooted planar triangulations, II : the case  $\lambda = \tau + 1$
- [Tutte 73] Chromatic sums for rooted planar triangulations, III : the case  $\lambda = 3$
- [Tutte 73] Chromatic sums for rooted planar triangulations, IV : the case  $\lambda = \infty$
- [Tutte 74] Chromatic sums for rooted planar triangulations, V : special equations
- [Tutte 78] On a pair of functional equations of combinatorial interest
- [Tutte 82] Chromatic solutions
- [Tutte 82] Chromatic solutions II
- [Tutte 84] Map-colourings and differential equations



- [Tutte 95]: Chromatic sums revisited

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### Theorem [Tutte]

- For  $q = 2 + 2 \cos \frac{2\pi}{m}$ ,  $q \neq 4$ , the series  $T(1, y) \equiv T(t; 1, y)$  satisfies a polynomial equation with one catalytic variable  $y$ .

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- When  $q$  is generic, the generating function of properly  $q$ -coloured planar triangulations is differentially algebraic:

$$2q^2(1-q)t + (qt + 10H - 6tH')H'' + q(4-q)(20H - 18tH' + 9t^2H'') = 0$$

with  $H(t) = t^2T_2(q, \sqrt{t}; 1)/q$ .



## Adapt this to other equations!

- [Tutte 73] Chromatic sums for rooted planar triangulations: the cases  $\lambda = 1$  and  $\lambda = 2$
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[Tutte 95]: Chromatic sums revisited

## Our results

- Let  $M(q, \nu, t; x, y)$  be the Potts generating function of planar maps:

$$M(x, y) \equiv M(q, \nu, t; x, y) = \frac{1}{q} \sum_M Z_M(q, \nu) t^{e(M)} x^{\text{dv}(M)} y^{\text{df}(M)},$$

where  $\text{dv}(M)$  (resp.  $\text{df}(M)$ ) is the degree of the root-vertex (resp. root-face).

### Theorem

- For  $q = 2 + 2 \cos \frac{j\pi}{m}$ ,  $q \neq 0, 4$ , the series  $M(q, \nu, t; 1, y) \equiv M(1, y)$  satisfies a polynomial equation with one catalytic variable  $y$ , and the complete Potts generating function  $M(q, \nu, t; x, y)$  is algebraic.
- When  $q$  is generic,  $M(q, \nu, t; 1, 1)$  is differentially algebraic:

(an explicit system of differential equations)

[mbm-Bernardi 09] Counting colored planar maps: algebraicity results. Arxiv:0909:1695

[mbm-Bernardi 12] Counting colored planar maps: differential equations

## Algebraicity via Tutte's invariants method: From two to one catalytic variables

- The equation is linear in  $M(x, y)$ :

$$\begin{aligned} M(x, y) = & 1 + xyt((\nu - 1)(y - 1) + qy) M(x, y)M(1, y) \\ & + xyz t(x\nu - 1)M(x, y)M(x, 1) \\ & + xyt(\nu - 1) \frac{xM(x, y) - M(1, y)}{x - 1} + xyz t \frac{yM(x, y) - M(x, 1)}{y - 1}. \end{aligned}$$

Write it

$$K(x, y)M(x, y) = R(x, y)$$

where  $K$  (the **kernel**) and  $R$  involve  $x, y, M(x, 1), M(1, y)$  (and  $q, \nu, t$ ).

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- **Cancel the kernel** by specializing  $y$ : There exists two (non-explicit) series  $Y_1(x)$  and  $Y_2(x)$  such that

$$K(x, Y_i) = 0 \quad (\text{which implies } R(x, Y_i) = 0)$$

We have 4 equations relating  $x, M(x, 1),$  the  $Y_i$  and  $M(1, Y_i)$ .

## Algebraicity via Tutte's invariants method (cont'd)

- From these 4 equations, **eliminate**  $x$  and  $M(x, 1)$ : this leaves 2 equations between the  $Y_i$  and  $M(1, Y_i)$ , which (when  $q = 2 + 2 \cos(j\pi/m)$ ) can be written

$$\begin{aligned} I(Y_1) &= I(Y_2) && \text{with } I(y) = tyqM(1, y) + \frac{y-1}{y} + \frac{ty}{y-1} \\ J(Y_1) &= J(Y_2) && \text{where } J \text{ involves the Chebyshev polynomial } T_m \end{aligned}$$

The series  $I(y)$  and  $J(y)$  are **invariants**.

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The series  $I(y)$  and  $J(y)$  are **invariants**.

- **The theorem of invariants**: every invariant is a polynomial in the elementary invariant  $I(y)$ . Hence there exists series  $F_0, \dots, F_m$  (depending on  $q, \nu$  and  $t$  only) such that

$$J(y) = \sum_{i=0}^m F_i I(y)^i \tag{2}$$

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$$J(y) = \sum_{i=0}^m F_i I(y)^i \tag{3}$$

- The series  $I(y)$  and  $J(y)$  involve  $y$  and  $M(1, y)$ , while the (unknown) series  $F_i$  are independent from  $y$ . Hence (3) is an equation with one catalytic variable ( $y$ ) satisfied by  $M(1, y)$ .

## Algebraicity via Tutte's invariants method: Examples

- For  $q = 1$  and  $M(y) \equiv M(1, y)$ , one obtains

$$M(y) = 1 + y^2 t \nu M(y)^2 + \nu t y \frac{y M(y) - M(1)}{y - 1}.$$

- For  $q = 2$  (the Ising model),

$$\begin{aligned} & 4 t^2 y^4 \nu (y - 1)^2 (\nu + 1) M(y)^3 + 4 (y - 1) y^4 \nu (\nu + 1) t^2 M(y)^2 \\ & \quad + (y - 1)^2 y^2 (\nu^2 y - y - \nu^2 - 6 \nu - 1) t M(y)^2 + \nu y^4 (\nu + 1) t^2 M(y) \\ & + y^2 (y - 1) (\nu^2 y + 3 \nu y - \nu^2 - 6 \nu - 1) t M(y) - (y - 1)^2 (\nu y - y - \nu - 1) M(y) \\ & \quad - 2 t^2 y^3 \nu (y - 1) (\nu + 1) M(1) M(y) - 2 t^2 y^2 \nu (\nu + 1) (y - 1) M(1)^2 \\ & \quad - y^2 \nu (2 y - 1) (\nu + 1) t^2 M(1) - y (y - 1) (-y + \nu^2 y - 2 \nu y - \nu - \nu^2) t M(1) \\ & \quad - \nu t^2 y^2 (\nu + 1) (y - 1) M'(1) + (y - 1)^2 (\nu y - y - \nu - 1) = 0 \end{aligned}$$

- For  $q = 3$ , one obtains a big equation involving  $M(y)$ ,  $M(1)$ ,  $M'(1)$  and  $M''(1)$ .

Equations with one catalytic variable  $\Rightarrow$  algebraic solutions



## Example: The Ising model on planar maps ( $q = 2$ )

Let  $A$  be the series in  $t$ , with polynomial coefficients in  $\nu$ , defined by

$$A = t \frac{(1 + 3\nu A - 3\nu A^2 - \nu^2 A^3)^2}{1 - 2A + 2\nu^2 A^3 - \nu^2 A^4}.$$

Then the Ising generating function of planar maps is

$$M(2, \nu, t; 1, 1) = \frac{1 + 3\nu A - 3\nu A^2 - \nu^2 A^3}{(1 - 2A + 2\nu^2 A^3 - \nu^2 A^4)^2} P(\nu, A)$$

where

$$P(\nu, A) = \nu^3 A^6 + 2\nu^2(1 - \nu)A^5 + \nu(1 - 6\nu)A^4 \\ - \nu(1 - 5\nu)A^3 + (1 + 2\nu)A^2 - (3 + \nu)A + 1.$$

↪ **Asymptotics:** Phase transition at  $\nu_c = \frac{3 + \sqrt{5}}{2}$ , critical exponents...

## Example: properly 3-coloured planar maps ( $q = 3, \nu = 0$ )

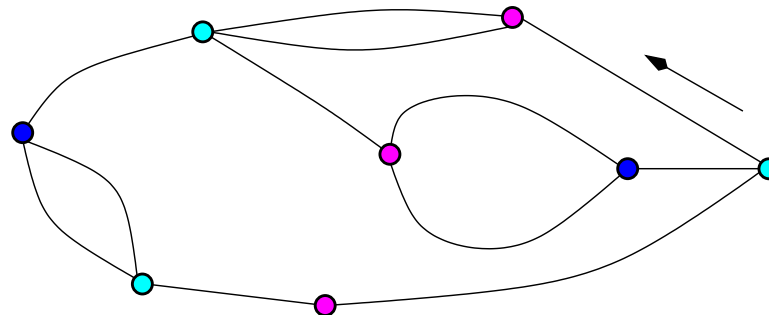
Let  $A$  be the quartic series in  $t$  defined by

$$A = t \frac{(1 + 2A)^3}{(1 - 2A^3)}.$$

Then the generating function of properly 3-coloured planar maps is

$$M(3, 0, t; 1, 1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}$$

↪ **Asymptotics:** A random loopless planar map with  $n$  edges has approximately  $(1.42\dots)^n$  proper 3-colourings



## Our results: when $q$ is generic

- Let  $M(q, \nu, t; x, y)$  be the Potts generating function of planar maps:

$$M(x, y) \equiv M(q, \nu, t; x, y) = \frac{1}{q} \sum_M Z_M(q, \nu) t^{e(M)} x^{\text{dv}(M)} y^{\text{df}(M)},$$

where  $\text{dv}(M)$  (resp.  $\text{df}(M)$ ) is the degree of the root-vertex (resp. root-face).

### Theorem

- For  $q = 2 + 2 \cos \frac{j\pi}{m}$ ,  $q \neq 4$ , the series  $M(q, \nu, t; 1, y) \equiv M(1, y)$  satisfies a polynomial equation with one catalytic variable  $y$ , and the complete Potts generating function  $M(q, \nu, t; x, y)$  is algebraic.
- When  $q$  is generic,  $M(q, \nu, t; 1, 1)$  is differentially algebraic:

(an explicit system of differential equations)

## An explicit system of differential equations

Let  $D(t, v) = (q\nu + (\nu - 1)^2) v^2 - q(\nu + 1)v + (q + t(\nu - 1)(q - 4)(q + \nu - 1))$ .

- There exists a unique 8-tuple  $(P_0(t), \dots, P_3(t), Q_0(t), Q_1(t), R_0(t), R_1(t))$  of series in  $t$  with polynomial coefficients in  $q$  and  $\nu$  such that

$$\frac{1}{R} \frac{\partial}{\partial v} \left( \frac{R^2}{PD^2} \right) + \frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = 0,$$

where

$$P(t, v) = v^4 + P_3(t)v^3 + P_2(t)v^2 + P_1(t)v + P_0(t),$$

$$Q(t, v) = v^2 + Q_1(t)v + Q_0(t),$$

$$R(t, v) = (q + \nu - 3)v^2 + R_1(t)v + R_0(t),$$

with the initial conditions (at  $t = 0$ ):

$$P(0, v) = v^2(v - 1)^2 \quad \text{and} \quad Q(0, v) = v(v - 1).$$

## An explicit system of differential equations (cont'd)

- The Potts generating function of planar maps,  $M(1, 1) \equiv M(q, \nu, t; 1, 1)$ , satisfies

$$12t^2 (q\nu + (\nu - 1)^2) M(q, \nu, t; 1, 1) = 8t(q + \nu - 3)Q_1(t) - Q_1(t)^2 + P_2(t) - 2Q_2(t) - 4t(2 - 3\nu - q) - 12t^2 (q + \nu - 3)^2.$$

### Questions

1. Use the structure of

$$\frac{1}{R} \frac{\partial}{\partial \nu} \left( \frac{R^2}{PD^2} \right) + \frac{1}{Q} \frac{\partial}{\partial t} \left( \frac{Q^2}{PD^2} \right) = 0,$$

to obtain a single differential equation (or an expression?) for  $M(q, \nu, t; 1, 1)$ .

2. Relate this to elliptic functions, and to the papers of Bonnet & Eynard, and Guionnet, Jones, Shlyakhtenko & Zinn-Justin

## An analogous system for triangulations

Let  $D(t, v) = q\nu^2v^2 + (\nu - 1)(4(\nu - 1) + q)v + (q\nu(\nu - 1)(q - 4)t + (\nu - 1)^2)$ .

- There exists a unique 7-tuple  $(P_0(t), \dots, P_2(t), Q_0(t), Q_1(t), R_0(t), R_1(t))$  of series in  $t$  with polynomial coefficients in  $q$  and  $\nu$  such that

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where

$$P(t, v) = v^3 + P_2(t)v^2 + P_1(t)v + P_0(t),$$

$$Q(t, v) = 2\nu v^2 + Q_1(t)v + Q_0(t),$$

$$R(t, v) = R_1(t)v + R_0(t),$$

with the initial conditions (at  $t = 0$ ):

$$P(0, v) = v^2(v + 1/4) \quad \text{and} \quad Q(0, v) = v(2\nu v + 1).$$

- Expression of the Potts GF of triangulations in terms of the  $P_i$  and  $Q_i$

## Special cases for triangulations

- Properly coloured triangulations ( $\nu = 0$ ): Tutte's DE of order 2
- The Potts model with 4 states ( $q = 4$ ): DE of order 2
- Spanning forests on (near-)cubic maps ( $q = 0$ ): DE of order 2

## Many questions are left...

### A. Equations with two catalytic variables

- Elimination in the systems of differential equations
- Connections with elliptic functions, and with [Eynard-Bonnet 99] and [Guionnet et al. 10]



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- Elimination in the systems of differential equations
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### B. More combinatorics

- Understand algebraic series, e.g., for 3-coloured planar maps:

$$M(3, 0, t; 1, 1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2} \quad \text{with} \quad A = t \frac{(1 + 2A)^3}{(1 - 2A^3)}$$

- Understand differential equations, e.g., for properly  $q$ -coloured triangulations:

$$2q^2(1 - q)t + (qt + 10H - 6tH')H'' + q(4 - q)(20H - 18tH' + 9t^2H'') = 0$$

(with J. Courtiel: maps with a spanning forest ( $q = 0$ ))

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### C. Asymptotics

- Asymptotic number of properly  $q$ -coloured maps?

(done for triangulations  $q \in (28/11, 4) \cup [5, \infty)$  [Odlyzko-Richmond 83])

- More generally, phase transitions and critical exponents of the Potts model

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- The Potts model with 4 states ( $q = 4$ ): differential equation of order 2
- Spanning forests on (near-)cubic maps ( $q = 0$ ): differential equation of order 2

$$\begin{aligned} & \left( 3t\beta^4 T'^4 - \beta^3 (t - \beta t + 5\beta T) T'^3 + 4\nu (t - \beta t + 5\beta T)^2 \right) T'' \\ & - 48t\beta^2 (1 + \beta) T'^3 + 8\beta\nu (t - \beta t + 5\beta T) T'^2 \\ & + 4(\beta^2 - 1)(t - \beta t + 5\beta T) T' = 0 \end{aligned}$$

with  $\beta = \nu - 1$