Advanced Graph Theory
(4TIN917U)

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Lecture 1: Introduction.

Around this course:
- There is a weekly graph seminar (Fridays 2pm to 3pm). You are welcome to attend (or watch the recordings).
  Obviously no impact on your evaluation.
- There is a mentoring program (having 1-1 meetings with PhD students from LaBRI). Email us if you'd like to register!
- Interrupt us if anything is unclear.

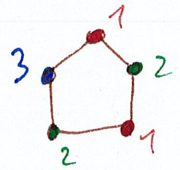
The grading will be one homework (8 pages on a topic among a list) and one final exam. Lecture notes will be allowed for the exam.

This course will contain some things you need to understand (denoted ⚠) and some things that are just for general knowledge (denoted ✱). General knowledge ✱ will not feature in the exam. We will generally not give names/pointers for ✱.

The first lecture is introductory. Following ones will be more involved, don't worry if you don't learn much today.

⚠ A graph $G=(V,E)$ is a set of vertices V with a (possibly empty) set of edges $E \subseteq V \times V$. Two vertices are adjacent if the corresponding pair belongs to E.

⚠ $\forall k \in \mathbb{N}$, $\forall$ graph $G=(V,E)$, G admits a k-colouring if $\exists c : V(G) \to [\![1,k]\!]$ such that $\forall uv \in E$, $c(u) \neq c(v)$.

Example: 

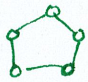
is a 3-colouring of .

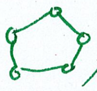
⚠ The chromatic number of G, denoted $\chi(G)$, is the smallest k such that G admits a k-colouring.

Example: $\chi($$) = 3$ Example: $\chi(G) \leq 2 \Leftrightarrow G$ is bipartite

The chromatic number of a graph helps solve scheduling issues (eg variables) and generally ressource attribution.

The stability number of a graph G, denoted $\alpha(G)$, is the maximum size of a stable set in G. A stable set $S \subseteq V(G)$ is a set such that $\forall u, v \in S$, $uv \notin E(G)$.

Example: 11 distinct stable sets in 

$\alpha($  $) = 2$

Given a k-colouring c of a graph G, the vertices with colour 1 induce a stable set. A k-colouring is a partition of the vertices in k stable sets.

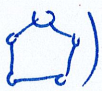
Example: How many 3-colourings of  ?

For any graph G, we have $\chi(G) \geq \dfrac{|V(G)|}{\alpha(G)}$

Question: Find a graph G with $\chi(G)$ much larger than $\dfrac{|V(G)|}{\alpha(G)}$?

The clique number of a graph G, denoted $\omega(G)$, is the maximum size of a clique in G. A clique $C \subseteq V(G)$ is a set such that $\forall u, v \in C$ with $u \neq v$, $uv \in E(G)$.

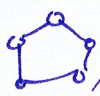
Example: How many cliques in  ?

$\omega($  $) = 2$

Upper bounds

For any graph G, we have $\chi(G) \leq |V(G)|$.

The maximum degree of a graph G, denoted $\Delta(G)$, is the maximum number of neighbours a vertex can have. Two vertices are neighbours if they are adjacent.

Example: $\Delta($  $) = 2$

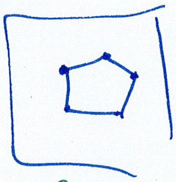
Any greedy algorithm yields $\chi(G) \leq \Delta(G) + 1$.

This is tight in particular for 

⚠ **Theorem (Brooks 1964):**

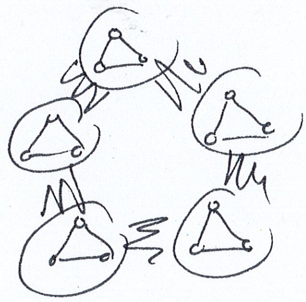
For any connected graph G that is neither a clique nor an odd cycle, $\chi(G) \leq \Delta(G)$.

⚠ A graph is connected if any two vertices are connected by a path.

Example:  is connected.  is not connected.  is

* **Conjecture** (Borodin, Kostochka 1977)

For any graph G, if $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta(G)-1$, we have $\chi(G) \leq \Delta(G)-1$.

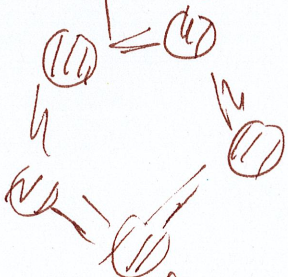
* The hypothesis "$\Delta(G) \geq 9$" is necessary. See 

* The conjecture holds for $\Delta(G) \geq 10^6$.

* Conjecture (Reed 1998).

For any graph G, we have $\chi(G) \leq \left\lceil \frac{(\Delta(G)+1) + \omega(G)}{2} \right\rceil$

* Question: What is the largest p such that for any graph G, $\chi(G) \leq \lceil (1-p)\cdot(\Delta(G)+1) + p\cdot\omega(G) \rceil$?

$p \geq 0$. $p \leq \frac{1}{2}$ because .

* Best known: $p \geq \frac{1}{3}$ for $\Delta(G)$ large enough (used to be $10^{-7}, \frac{1}{26}, \frac{1}{13}, ..$)

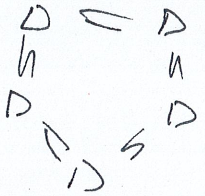
Usually, tool behind large numbers: $\boxed{\text{Probabilistic Method}}$

(Option 1)

Graphs with high chromatic number

Question: Does every graph G satisfy $\chi(G) \leq \omega(G)+1$?

Answer: No



Question: Is there a function f s.t. for every graph G, $\chi(G) \leq f(\omega(G))$?

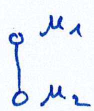
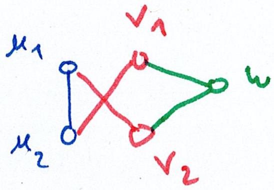
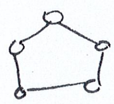
Answer: still no. ($\hookrightarrow$ Sometimes yes $\boxed{\chi\text{-bounded graph classes}}$)

Mycielski construction:

Graph $G=(V,E)$. Its Mycielski $M(G) = (V', E')$ is defined as $V' = \{u_i, v_i \mid 1 \leq i \leq |V|\} \cup \{w\}$, where $V = \{u_i \mid 1 \leq i \leq |V|\}$.

$$\forall i,j \begin{cases} u_i u_j \in E' \text{ iff } u_i u_j \in E. \\ u_i v_j \in E' \text{ iff } u_i u_j \in E \end{cases}$$

and $\forall i$, $v_i w \in E'$.

Example. $G =$  $M(G) =$  $=$ 

Observation: If $\omega(G) = 2$ then $\omega(M(G)) = 2$

Lemma: For any graph G, we have $\chi(M(G)) = \chi(G)+1$

Proof: $\chi(M(G)) \leq \chi(G)+1$ (easy).

Assume for a contradiction that $M(G)$ admits a $\chi(G)$-colouring c

Assume wlog that $c(w) = \chi(G)$. Then $c(\{v_1, \dots, v_{|V|}\}) \subseteq \{1, \dots, \chi(G)-1\}$

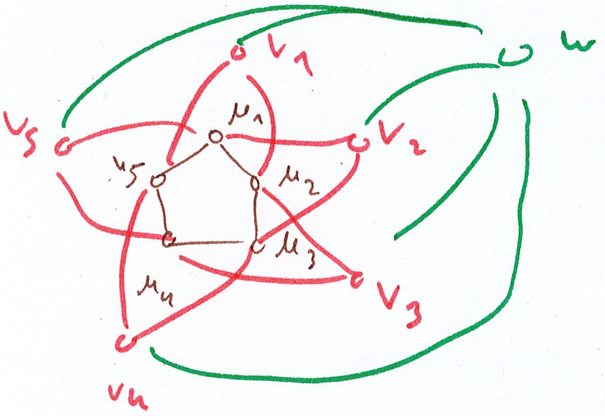
We obtain a $\chi(G)-1$ colouring of $M(G)[\{u_1, \dots, u_{|V|}\}] = G$ by considering the function ~~$(\chi(G)-1)$-colouring~~ c' obtained from c as follows:

$\forall i$, if $c(u_i) \leq \chi(G)-1$ then $c'(u_i) = c(u_i)$

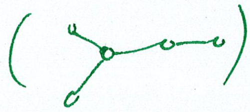
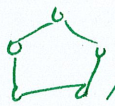
if $c(u_i) = \chi(G)$ then $c'(u_i) = c(v_i)$.

Note that c' is a $(\chi(G)-1)$-colouring of G, a contradiction.

Second example :



⚠ The girth $g(G)$ of a graph G is the length of a smallest cycle in G.

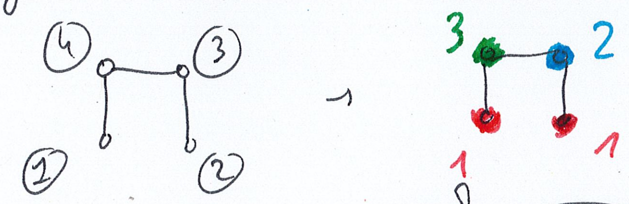
$g($  $) = +\infty$ $g($  $) = 5$

Examples of graphs with high $g(G)$ and high $\chi(G)$?

↳ Probabilistic method again.

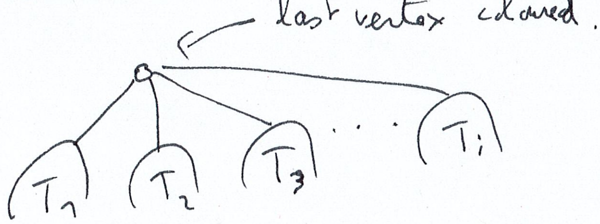
Greedy colouring

⚠ Example of a graph with some ordering such that a greedy algorithm gives a non-optimal solution?

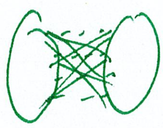


* Build by induction on k a tree (→ connected graph w/ no cycle) with some ordering such that a greedy algorithm ~~gi~~ uses k colours.

T_1 : o T_{i+1} :



* other example :



⚠ Subgraph, induced subgraph. → Cographs.

Option 2: Graph Minors

Option 3: Edge Colouring