

# List Coloring

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## 1 Introduction

List coloring is a variant of proper coloring, it was invented independently by Vizing [Viz76], and by Erdős, Rubin and Taylor [ERT79] in the 70'.

**⚠ Definition 1** (list coloring).

Let  $G = (V, E)$  be a graph, and  $L$  be a list assignment of  $G$ , ie a mapping that maps each vertex of  $G$  to a list of integers. The graph  $G$  is called  $L$ -colorable, if there exists a proper coloring  $\varphi$  of the vertices s.t. for each vertex  $v$ ,  $\varphi(v) \in L(v)$ .

**⚠ Definition 2.** The minimum integer  $k$  s.t.  $G$  is  $L$ -colorable for any list assignement  $L$  s.t. for any vertex  $v$ ,  $|L(v)| = k$  is called the choosability number of  $G$ , and is denoted by  $ch(G)$  (or  $\chi_l(G)$ ).

Similarly to the chromatic number  $\chi(G)$ , the question is usually to minimize  $ch(G)$ .

**Example 3.**

- $ch(C_{2p+1}) = 3$ ,
- $ch(K_k) = k + 1$ ,
- $ch(G) \leq \Delta(G) + 1$ ,
- $ch(G) \leq k + 1$  if  $G$  is  $k$ -degenerate,
- $ch(G) \leq 5$  if  $G$  is planar (see section 3),
- $ch(L(G)) = \chi(L(G))$  if  $G$  is bipartite (see section 4).

### 1.1 General bounds

As proper coloring is just a special case of list coloring (where all the lists are the same for all the vertices), it is clear that:

**⚠**  $ch(G) \geq \chi(G) \geq \omega(G)$ .

Moreover, using a greedy algorithm, we also have that:

**⚠** If  $G$  is  $k$ -degenerate, then  $ch(G) \leq k + 1$ .

(recall that: a graph is  $k$ -degenerate if all its subgraphs have a vertex of degree at most  $k$ ).

(\*) Alon proved in 2000 [Al00] that the choice number can also be bounded from below as a function of the minimum degree  $\delta(G)$ .

(\*)**Theorem 1** (Alon).  $ch(G) \leq (1/2 + o(1)) \cdot \log(\delta(G))$ .

Moreover, Erdős, Rubin and Taylor [ERT79] proved that  $ch(K_{\delta, \delta}) = (1 + o(1)) \cdot \log(\delta)$  (the complete bipartite graph on  $2 \cdot \delta$  vertices). So the bound of Alon is tight up to a factor 2. The proof of Alon uses a probabilistic method called the containers method.

*Sketch of proof.* The proof uses the probabilistic method. We have a graph  $G$  with minimum degree  $\delta$ , and an integer  $s$  s.t.  $d > \frac{4 \cdot (s^2 + 1)^2}{\log_2(e)^2} \cdot 2^{2s}$ . The principle is to find a list assignment  $L$  with lists of size  $s$ , with color set  $\{1, \dots, s^2\}$  s.t  $G$  does not have a  $L$ -coloring. We consider a set  $S$  of vertices where  $S$  is build as follows: we pick each vertex uniformly at random with probability  $\frac{1}{\sqrt{\delta}}$ . Then we prove that at least half of the vertices of the graph are good ie:

- they are not in  $S$
- for any set of colors  $C$  containing half of the available colors (the containers), a good vertex has a neighbor  $u$  in  $S$  with  $L(u) \subset C$ .

So if  $\varphi_S$  is a partial  $L$ -coloring of  $G$  where only the vertices which are not in  $S$  are colored, each good vertex will see strictly more than half of the colors, so it has strictly less than half of the colors available to it. This, combined the fact that we have at least half of the vertices which are good guarantees that there is a list assignment  $L$  s.t.  $G$  is not  $L$ -colorable. □

## 1.2 Arbitrary large gap between $ch(G)$ and $\chi(G)$

One idea would be to try to find an upper bound of  $ch(G)$  as a function of  $\chi(G)$ , however, such a function does not exist. We can build a family  $\mathcal{G}$  of bipartite graph, (so for any  $G \in \mathcal{G}$ ,  $\chi(G) = 2$ ), s.t. for any integer  $k$ , there exists  $g \in \mathcal{G}$  with  $ch(G) > k$ , ie, there exists a list assignment  $L$  s.t. the sizes of the lists are  $k$ ,  $G$  is not  $L$ -colorable.

### 1.2.1 ▲ Construction of $\mathcal{G}$

Let  $k$  be an integer, we will build a bipartite graph  $G$  s.t.  $G$  is not  $L$ -colorable for lists of size  $k$ . Let us consider the complete bipartite graph  $K_{k,k}$ , and let us call  $X = \{x_0, \dots, x_{k-1}\}$  the part with  $k$  vertices and  $Y$  the other part. We now have to assign the list to the vertices, the lists will be subsets of the set of colors  $\{0, \dots, k^k - 1\}$ . For each  $i$ , each vertex  $x_i$  will be assigned the list  $L(x_i) = \{i \times k, i \times k + 1, \dots, i \times k + k - 1\}$ , and for each vertex  $y$  of  $Y$  we assign a list  $L(y) = \{c_0, \dots, c_{k-1}\}$  where for any  $i$ ,  $c_i \in L(x_i)$ , and the lists are pairwise disjoint (note that there are exactly  $k^k$  such distinct lists, so each  $y \in Y$  can be assigned a different list, and each different combination of colors appears exactly once).

*Proof.* ▲ We now prove that  $G$  is not  $L$ -colorable. Toward contradiction, assume that  $G$  has a  $L$ -coloring  $\varphi$ , and wlog, assume that  $\forall i \neq 0, \varphi(x_i) = i \times k$ . We now prove that there is no color available for  $x_0$ . For each  $i$ , let us denote by  $y_i$  the vertex of  $Y$  with the list  $\{i, k, 2k, \dots, (k-1) \times (k-1)\}$ . As each  $x_i$  is colored ( $i \times k$ ), and is connected to all the vertices of  $Y$ , each  $y_i$  has only one color available, each vertex  $y_i$  has to be colored with color  $i$ . So the vertex  $x_0$  is connected to vertices with every color in the range  $\{0, \dots, k-1\}$ , so it does not have an available color; a contradiction. □

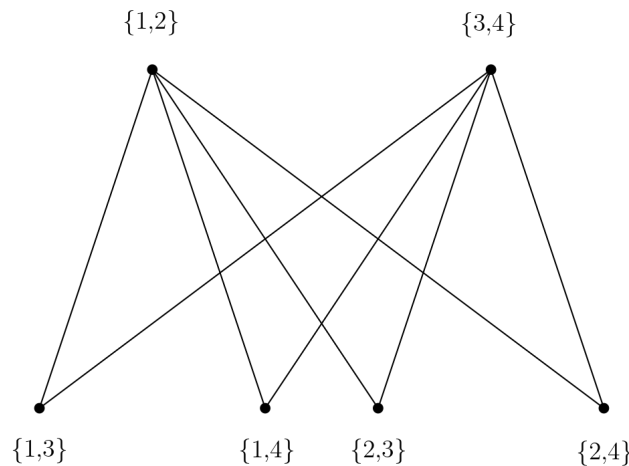


Figure 1: The construction for  $k = 2$ .

## 2 An analog of Brooks theorem

We can also bound the choice number with a function of the degree of the vertices, in this section we will see an analog of the Brooks theorem for list coloring.

**Definition 4** (degree-list assignment). A list assignment  $L$  is called a degree-list assignment, if for every vertex  $v$ , we have  $|L(v)| = d(v)$  where  $d(v)$  is the degree of  $v$ .

Gallai proved that any graph with a degree-list assignment  $L$  is  $L$ -colorable unless it is a Gallai forest. Actually the two exceptions of the Brooks theorem appear in the definition of a Gallai forest.

**Definition 5** ( $k$ -connectivity). A graph  $G$  is  $k$ -connected, if one needs to remove at least  $k$  vertices of  $G$  to disconnect it.

**Definition 6** (block decomposition). A block is a maximal 2-connected component of a graph, ie a maximal component with no cut vertex. Let  $G$  be a graph, if we merge each block of  $G$  into one vertex, we obtain a tree  $T$  where each vertex of  $T$  represents a block of  $G$ . This tree  $T$  is called the block decomposition of the  $G$ .

**Definition 7** (Gallai tree). A graph  $G$  is called a Gallai tree if each block is an odd cycle or a clique. A Gallai forest is a graph where each connected component is a Gallai tree.

We are now ready to state the theorem.

**Theorem 1** (Gallai). Let  $G$  be a connected graph and  $L$  a degree-list assignment, then  $G$  is  $L$ -colorable unless  $G$  is a Gallai tree.

In order to prove the theorem, we first need to prove this weaker Lemma.

**Lemma 8** (Erdős & al). Let  $G$  be a connected graph, and let  $L$  be a list assignment s.t.:

- $\forall u, |L(u)| \geq d_G(u)$  and
- $\exists v$  s.t.  $|L(v)| > d_G(v)$ .

then  $G$  is  $L$ -colorable.

*Proof.* Let  $G$  be a minimum counterexample, and let  $v$  be the vertex with  $|L(v)| > d_G(v)$ . Let us consider the graph  $G' = G \setminus \{v\}$ . In each component  $G_i$  of  $G'$  we have a vertex  $u$  that was a neighbor of  $v$  in  $G$ , so in  $G_i$ , we have that  $|L(u)| > d_{G_i}(u)$ , and  $|L(w)| \geq d_{G_i}(w)$  for any other vertex  $w$ . By the minimality of  $G$ , we can find a  $L$ -coloring of each  $G_i$ . As  $|L(v)| > d_G(v)$  we still have one available color for  $v$ , and so we can extend this  $L$ -coloring of the  $G_i$  to a  $L$ -coloring of  $G$ .  $\square$

We are now ready to prove the theorem.

*Proof of Gallai's theorem.* Let  $G$  be a minimal counterexample along with a degree-list assignment  $L$ , we need to distinguish whether  $G$  is 2-connected nor not.

*Case 1* ( $G$  is 2-connected). If  $G$  is 2-connected, we will prove that  $G$  is  $L$ -colorable unless every vertex of  $G$  has the same list; as it is a degree-list assignment it would mean that  $G$  is  $L$ -colorable unless  $G$  is a  $k$ -regular graph which is not  $k$ -colorable: by Brooks theorem the only two exceptions are odd cycles and cliques.

Toward contradiction, assume that two vertices  $x$  and  $y$  have two different lists. If two such vertices exist, then two adjacent vertices  $x$  and  $y$  have different lists. Let  $\alpha$  be a color of  $L(y)$  which does not appear in  $L(x)$  (such a color always exists up to inverting the role of  $x$  and  $y$ ). We consider a new assignment  $L'$  where we remove the color  $\alpha$  from every neighbor of  $y$  (this does not change the list of  $x$  by definition of  $\alpha$ ). We now consider the graph  $G' = G \setminus \{y\}$ , note that as  $G$  is 2-connected,  $G'$  is a connected graph. For each vertex  $u$  of  $G'$  which is not  $x$  we have  $|L'(u)| \geq d_{G'}$ , and for  $x$ , we have that  $|L(u)| > d_{G'}(u)$ , so by the previous Lemma, we can find a  $L'$ -coloring of  $G'$ . It suffices to give the color  $\alpha$  to  $y$  to get a  $L$ -coloring of  $G$ . So each vertex in  $G$  has the same list.

*Case 2* ( $G$  has a cut vertex). We consider the block decomposition  $T$  of  $G$ . As  $T$  is a tree, it has at least two leaves  $B_1, B_2$  which are blocks of  $G$ . Let  $x_1$  be a vertex of  $B_1$  which is not a cut vertex of  $G$  (unless  $|B_1| = 1$ , such a vertex always exists as  $B_1$  is 2-connected), and let  $\alpha$  be a color of  $L(x_1)$ . We consider a new list-assignment  $L'$  where we remove the color  $\alpha$  from the list of the neighbors of  $x_1$ , and we consider the graph  $G' = G \setminus \{x_1\}$ . The graph  $G'$  is connected, and for every vertex  $v$ ,  $|L'(v)| = d(v)$ , so if  $G'$  is not a Gallai tree, by the minimality of  $G$ , we can find a list coloring of  $G'$  which is easily extendable to  $G$  by giving the color  $\alpha$  to  $x_1$ . If  $G'$  is a Gallai tree, it means that  $B_1$  is the only block of  $G$  which is not a complete graph or an odd cycle, and that  $B_1 \setminus \{x_1\}$  is a complete graph or an odd cycle; it suffices to take a vertex  $x_2$  from  $B_2$ , and we have that  $G'' = G \setminus \{x_2\}$  which is not a Gallai tree.  $\square$

### 3 Planar graphs

A graph is planar if it can be drawn on the plane without edge crossing. One of the most famous result in graph theory, and especially in graph coloring is the 4-color theorem. This theorem guarantees that, to properly color the vertices of a planar graph, 4 colors always suffice.

The question of the choosability of planar graphs remained an open questions for decades. It was conjectured that planar graphs are 5-choosable, and it took 20 years to Thomassen [Tho94] to find an easy and elegant proof to this theorem, independent from the previous proof of the 5-color theorem (a planar graph is always 5-colorable).

**⚠ Theorem 2** (Thomassen). *Let  $G$  be a planar graph,  $ch(G) \leq 5$ .*

The proof is done by induction on the number of vertices, however, to prove the theorem, we actually prove this stronger Lemma.

**Lemma 9.** *Let  $G$  be a near triangulation and  $L$  a list assignment s.t.:*

- *For two special adjacent vertices  $u$  and  $v$  of the outerface, we have that  $|L(u)| = |L(v)|$  and  $L(u) \cap L(v) = \emptyset$ ,*
- *for any other vertex  $w$  of the outerface, we have that  $|L(w)| = 3$ ,*
- *for all the other vertices, we have a list of size 5.*

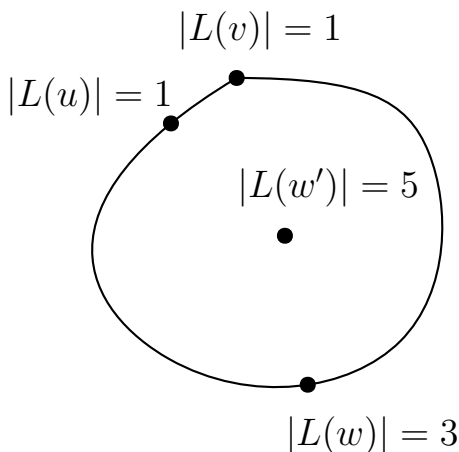


Figure 2: The graph and the sizes of the lists as stated in the Lemma

Before proving the Lemma, we prove that the theorem is a direct consequence of the Lemma.

*Proof.* Let  $G$  be a planar graph and  $L$  a 5-list assignment. We add to  $G$  edges until we obtain a triangulation  $G'$  where  $uvw$  is the outerface of  $G'$ . We then pick a color for  $u$  and  $v$  respectively in  $L(u)$  and  $L(v)$  and remove these colors from  $L(w)$ , and we can finally apply the Lemma to find a  $L$ -coloring of  $G'$  which is clearly a  $L$ -coloring of  $G$ .  $\square$

We now prove the Lemma.

*Proof of Lemma 9.* We first prove that  $G$  is 2-connected and that the outerface has no chord. Let  $G$  be a minimum counter example with respect to the number of vertices, along with a list assignment  $L$  as stated.

**Claim 1.**  *$G$  has no cut vertex*

*Proof.* Assume that  $G$  has a cut vertex  $w$ . Either  $w \in \{u, v\}$ , or  $u$  and  $v$  belong to the same component  $G_1$  of  $G \setminus \{w\} = \bigcup G_i$ . For any  $i$ , the  $G_i$  are smaller than  $G$ , moreover,  $G_1$  verifies the three properties of the Lemma, so we can find a  $L$ -coloring of  $G_1$ . We can then find a  $L$ -coloring of the other  $G_i$ , as only  $w$  is already colored (ie only  $w$  has a list of size 1), it suffices to pick any available color for a neighbor of  $w$  in each  $G_i$  to find a  $L$ -coloring of each  $G_i$ , so finally we have a coloring for  $G$ .  $\square$

**Claim 2.** *The outerface of  $G$  has no chord*

*Proof.* Assume that the outerface  $F_o = \{w_1, \dots, w_k\}$  of  $G$  has a chord  $w_i w_j$ . Similarly to the previous case,  $u$  and  $v$  are in the same component  $G_1$  of  $G \setminus \{w w'\} = G_1 \cup G_2$ . So by the minimality of  $G$  we can find a  $L$ -coloring of  $G_1$ . We can then find a  $L$ -coloring of  $G_2$ , as only  $w$  and  $w'$  are already colored in  $G_2$  (ie they have lists of size 1), so finally we obtain a coloring of  $G$ .  $\square$

We are now ready to finish the prove of the Lemma. Let us denote by  $F_o = \{u, v, w_2, \dots, w_k\}$  the outerface of  $G$ . As  $G$  is a near-triangulation and  $F_o$  has no chord, the neighbors of  $w_k$  form a path  $\{u = u_1, \dots, u_l = w_{k-1}\}$  from  $u$  to  $w_{k-1}$ . Let  $\alpha$  and  $\beta$  be two colors of  $L(w_k)$  which are not in  $L(u)$ , we now consider the graph  $G' = G \setminus \{w_k\}$ , and the list assignment  $L'$  defined as follows:

- $\forall s \notin \{u_1, \dots, u_{l-1}\}, L'(s) = L(s)$ ,
- $\forall s \in \{u_1, \dots, u_{l-1}\}, L'(s) = L(s) \setminus \{\alpha, \beta\}$

The graph  $G'$  is smaller than  $G$ , and as the outerface  $F'_o$  of  $G'$  is exactly  $\{u, v, w_2, \dots, w_{k-1} = u_l, u_{l-1}, \dots, u_2\}$ . Each vertex of  $F'_o$  has a list of size 3, except  $u$  and  $v$  which have lists of size 1, so by the minimality of  $G$ , we can find a  $L'$ -coloring of  $G'$ . We now have to find an available color for  $w_k$  to extend the coloring to  $G$ . The only neighbor of  $w_k$  that can be colored  $\alpha$  or  $\beta$  is  $w_{k-1}$  as  $L(w_{k-1})$  is the only list of the neighbors of  $w_k$  which still contains the colors  $\alpha$  and  $\beta$  in  $L'$ , so we are guaranteed to have an available color for  $w_k$ .  $\square$

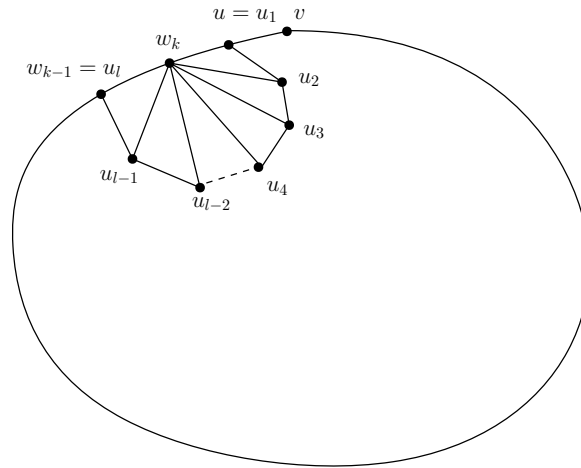


Figure 3: Illustration of the last case

So all planar graphs are 5-choosable, however, contrary to proper vertex coloring, not all planar graphs are 4-choosable. Mirzakhani discovered a planar graph which is not 4-choosable. She was in her last year of high school when she's discovered it, and not only this graph is relatively small and elegant, but it is 3-colorable. We won't discuss here how to build it, but you can have a teaser on the next Figure (see [Mar17]).

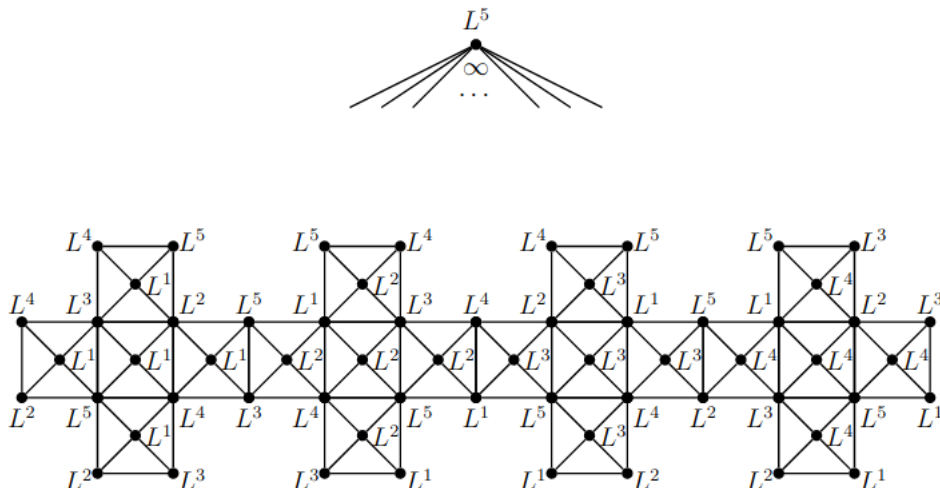


Figure 4: A planar graph with a list assignment such that the graph is not colorable: each list  $L^i$  contains all colors from 1 to 5, except color  $i$ . The top vertex is connected to every other vertex which is not in the center of one of the “squares”.

## 4 Edge-choosability

The notion of list coloring can also be generalized to edge-coloring. However, due to Vizing’s theorem it is easy to bound the list coloring index as a function of the chromatic index. Recall that we have  $\chi'(G) \leq \Delta(G) + 1$ , and that  $\Delta(L(G)) = 2\Delta(G) - 2$ , so we have that  $ch'(G) \leq 2\chi'(G)$ , where  $\chi'(G)$  and  $ch'(G)$  respectively denote the chromatic index, and the list chromatic index of the graph.

Dignitz conjectured that for any graph  $G$ ,  $ch'(G) = \chi'(G)$ , ie the fact that the edges only have specific available colors does not change anything to the coloring problem. This conjecture is still widely open, and has only been proven for a few classes of graphs, among them, bipartite graphs (by Galvin, see [Gal95]).

**△ Theorem 3** (Galvin). *Let  $G$  be a bipartite graph, then  $ch'(G) = \chi'(G)$ .*

The original statement of this theorem is about multigraphs, but we will only focus on the case of simple graphs here. The proof of this theorem relies on the following Lemma by Bondy, Boppana and Siegel cited by Alon and Tarsi [AT92] which uses an orientation of the graph to guarantee the existence of a list coloring. The use of graph orientation to find a coloring is pretty natural, for instance when applying a greedy algorithm to find a coloring, one just assign an orientation of the edges with respect to the order one has chosen to color the vertices (we orient each edge from the vertex which is not colored yet to the vertex already colored): we obtain an orientation where the outdegree of each vertex is at most  $\Delta$ . Similarly, for the case of  $k$ -degenerate graphs, we obtain an orientation where the outdegree of each vertex is at most  $k$ . The principle of this Lemma is the same: finding an orientation such that each outdegree is at most the size of the list.

But before stating the lemma, we need one more concept: a kernel of a directed graph, an analog of independent dominating set for directed graphs.

**△ Definition 10** (kernel). *Let  $G$  be a directed graph. A set  $K$  of  $V(G)$  is a kernel if:*

- $K$  is an independent set
- $\forall v \in V, v \in K$  or  $\exists u \in K$  s.t. there is an arc from  $v$  to  $u$ .

We are now ready to state the Lemma and to prove it.

**Lemma 11.** *Let  $G$  be a graph, and  $L$  a list assignment of  $G$ . If  $G$  has an orientation  $H$  s.t. for each vertex  $v$ ,  $d^+(v) < |L(v)|$ , and each induced subgraph of  $H$  has a kernel, then  $G$  is  $L$ -colorable.*

*Proof.* Let  $G$  be a minimum counterexample with a list assignment  $L$ , clearly  $|V(G)| > 0$ . Let us consider an orientation  $H$  of  $G$  as stated, and a color  $\alpha$  which appears at least once. We consider the graph  $H'$  induced by the vertices of  $H$  which have the color  $\alpha$  in their list, by hypothesis,  $H'$  has a kernel  $K'$ . We start by coloring

the vertices of  $K'$  with  $\alpha$ , and removing  $\alpha$  from the list of the other vertices of  $H'$ , we denote by  $L'$  this new list assignment. Since each vertex of  $H' \setminus K'$  sends an edge to a vertex of  $K'$ , we still have that  $d_{H' \setminus K'}^+(v) < L'(v)$ . But as  $H'' = H' \setminus K'$  is smaller than  $H$ , by the minimality of  $G$ , we can find a  $L'$ -coloring of  $H''$ , this coloring does not use the color  $\alpha$  by definition, so we can color the vertices of  $K'$  with the color  $\alpha$  to complete the coloring of  $G$ .  $\square$

Here is a sketch of the proof of Galvin's theorem.

*Sketch Galvin's proof.* Let  $G$  be a bipartite graph where  $V(G) = (X, Y)$  with  $X$  and  $Y$  the two parts of the bipartition of  $G$ , with a list assignment  $L$ . The principle of the proof is to find a good orientation  $H$  of  $L(G)$  s.t. each vertex has an outdegree less than the size of its list, and then to prove that each subgraph of  $H$  has a kernel. To do this, Galvin's uses a proper edge coloring of  $G$ , as  $G$  is bipartite, by Vizing's theorem [Viz64], we can obtain a  $\Delta$ -coloring  $\varphi$  of  $G$ . We now consider  $L(G)$ , the linegraph of  $G$ . Each pair of edge  $e, e'$  of  $G$  sharing a vertex  $v \in V(G)$  are adjacent in  $L(G)$ , we orient the edge  $ee'$  of  $L(G)$  according to  $\varphi$  as follows:

- $e \rightarrow e'$  if  $v \in X$  and  $\varphi(e) < \varphi(e')$ ,
- $e' \rightarrow e$  if  $v \in X$  and  $\varphi(e) > \varphi(e')$ ,
- $e \rightarrow e'$  if  $v \in Y$  and  $\varphi(e) > \varphi(e')$ ,
- $e' \rightarrow e$  if  $v \in Y$  and  $\varphi(e) < \varphi(e')$ .

Proving the bound on the outdegree is easy, and we leave as an exercise to the reader the "kernel part" of the proof.  $\square$

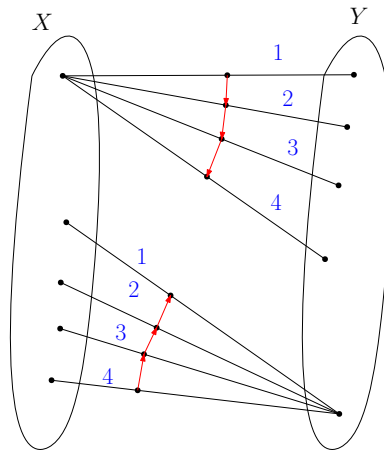


Figure 5: The edges of  $L(G)$  (in red) are oriented according to the colors of the edges of  $G$  (in blue).

## References

- [Alo00] Noga Alon. Degrees and choice numbers. *Random Structures & Algorithms*, 16(4):364–368, 2000.
- [AT92] Noga Alon and Michael Tarsi. Colorings and orientations of graphs. *Combinatorica*, 12(2):125–134, 1992.
- [ERT79] Paul Erdos, Arthur L Rubin, and Herbert Taylor. Choosability in graphs. In *Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, volume 26, pages 125–157, 1979.
- [Gal95] Fred Galvin. The list chromatic index of a bipartite multigraph. *Journal of Combinatorial Theory, Series B*, 63(1):153–158, 1995.
- [Mar17] William J Martin. On an early paper of maryam mirzakhani. *arXiv preprint arXiv:1709.07540*, 2017.

- [Tho94] Carsten Thomassen. Every planar graph is 5-choosable. *Journal of Combinatorial Theory, Series B*, 62(1):180–181, 1994.
- [Viz64] Vadim G. Vizing. On an estimate of the chromatic class of a p-graph. *Discret Analiz*, 3:25–30, 1964.
- [Viz76] Vadim G Vizing. Coloring the vertices of a graph in prescribed colors. *Diskret. Analiz*, 29(3):10, 1976.