

Graph Homomorphism

In general, a homomorphism on a certain structure is an application that preserves essential properties of that structure. For graphs, this means preserving the presence of edges.

Definition. \triangle A **homomorphism** is an application ϕ from a graph G to a graph H such that for all vertices u and v of G , $uv \in E(G) \Rightarrow \phi(u)\phi(v) \in E(H)$.

We denote the existence of a homomorphism from G to H by $G \rightarrow H$. A way to see a homomorphism from G to H is to give labels (numbers for instance) to the vertices of H , and give the number of a vertex $v \in H$ to every vertex u such that $\phi(u) = v$. Then we just have to make sure that for every edge in G , the corresponding vertices in H also share an edge.

Examples:

- If H is a subgraph of G , then $H \rightarrow G$ trivially.
- Every bipartite graph has a homomorphism to K_2 (just map every vertex of one of the two independent sets to the same vertex, and the other vertices to the other vertex of K_2).
- For every $n \in \mathbb{N}$, $C_{n+2} \rightarrow C_n$: from $v_1 \dots v_{n+2}$ to $u_1 \dots u_n$, we map v_i to u_i for $i \leq n$, v_{n+1} to u_{n-1} and v_{n+2} to u_n .

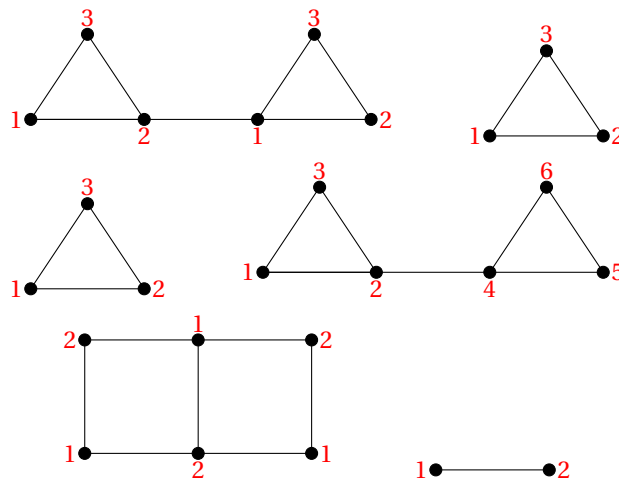


Figure 1: Exemples of isomorphisms. The graph on the left has a homomorphism to the graph on the right, defined as: every vertex is mapped to the vertex on the right with the same number.

Beware, the notion of homomorphism must not be mixed up with that of *isomorphism*.

Definition. \triangle An **isomorphism** from a graph G to a graph H is an application ϕ from $V(G)$ to $V(H)$ such that for all vertices u and v of G , $uv \in E(G) \Rightarrow \phi(u)\phi(v) \in E(H)$, and $u \neq v \wedge uv \notin E(G) \Rightarrow \phi(u) \neq \phi(v) \wedge \phi(u)\phi(v) \notin E(H)$.

We often do not differentiate between distinct isomorphic graphs, and only consider the graphs up to isomorphism. Two isomorphic graphs are essentially "the same" graphs.

Definition. \triangle An **endomorphism** is a homomorphism from a graph G to an induced subgraph H of G . If such an endomorphism exists, the induced subgraph H is called a **subgraph** of G .

Exemple: Any bipartite graph G with at least one edge uv has an endomorphism to $G[uv]$. $G[uv]$ is thus a subgraph of G .

Definition. \triangle An endomorphism that is an isomorphism, i.e. an isomorphism from a graph to itself, is an **automorphism**.

Exemples: On any graph, the identity is an endomorphism, and even an automorphism. A function on a cycle $v_0 v_1 v_2 v_3 v_4$ that maps for all i v_i to $v_{i+1[4]}$, is an automorphism. On a clique, any shuffling of the vertices is an automorphism.

△ Finding a homomorphism from a graph G to K_k is exactly the same thing as finding a proper k -coloring of G . Finding a homomorphism from K_k to G is exactly the same as finding a clique of size k in G .

△ \rightarrow is a *preorder* relation (it is reflexive and transitive).

Proof. **reflexivity:** the identity is a homomorphism, so for all G , $G \rightarrow G$.

transitivity: if ϕ is a homomorphism from G to H , and ψ is a homomorphism from H to Q , then $\psi \circ \phi$ is a homomorphism from G to Q : if $uv \in E(G)$, then $\phi(u)\phi(v) \in E(H)$, and thus $\psi(\phi(u))\psi(\phi(v)) \in E(Q)$. □

Property. △ If $G \rightarrow H$, then:

- $\omega(G) \leq \omega(H)$
- $\chi(G) \leq \chi(H)$

Proof. (*) This is a direct consequence of the fact that \rightarrow is transitive.

Since G has a clique of size $\omega(G)$, we know that $K_{\omega(G)} \rightarrow G$, and as $G \rightarrow H$, we also have $K_{\omega(G)} \rightarrow H$, so H has a clique of size $\omega(H)$, and thus $\omega(G) \leq \omega(H)$.

Since H is $\chi(H)$ -colorable, we know that $H \rightarrow K_{\chi(H)}$, and as $G \rightarrow H$, we also have $G \rightarrow K_{\chi(H)}$, so G is $\chi(H)$ -colorable, and $\chi(G) \leq \chi(H)$. □

Let \leftrightarrow be the associated equivalence relation, that is $G \leftrightarrow H$ if and only if $G \rightarrow H$ and $H \rightarrow G$. An *equivalence class* of this relation is a set of graphs that are all equivalent (that is a set S of graphs such that for any two graphs G and H in S , $G \leftrightarrow H$).

Definition. △ A **core** is a smallest element (in terms of number of vertices) in its equivalence class.

Lemma. △ Every equivalence class has a unique core up to isomorphism. That is, if two cores are equivalent, then they are isomorphic.

Proof. (*) Suppose G and H are cores and $G \leftrightarrow H$. Let ϕ be a homomorphism from G to H and ψ be a homomorphism from H to G . As G and H are cores, and thus minimal, the images of ϕ and ψ cannot be proper subsets of $V(H)$ and $V(G)$ respectively. They are bijections. Moreover, $\psi \circ \phi$ is a homomorphism from G to G . It maps each edge to an edge, and because the number of edges cannot increase, it also maps each non-edge to a non-edge, so it is an isomorphism. In particular ϕ cannot change the number of edges either, and is thus also an isomorphism. □

Definition. △ A **core** of a graph G is a subgraph of G that is a core.

Lemma. △ Every graph has a core.

Proof. (*) Let G be a graph, and H a core of the equivalence class of G . Since $G \leftrightarrow H$, in particular $H \rightarrow G$. Let W be the image of H by a homomorphism from H to G , and $G' = G[W]$. By definition of a homomorphism, G' has at most as many vertices as H . Moreover, $G' \leftarrow G$ since G' is a subgraph of G , and thus by transitivity $G' \leftarrow H$. But $H \leftarrow G'$ by definition of G' , so G' is a core. □

Lemma. △ A graph G is a core if and only if every endomorphism of G is an automorphism.

Proof. (*) \Rightarrow : Let ϕ be an endomorphism of G . Since G is a core, the image of ϕ cannot have fewer vertices than G , so it must be G itself, and thus ϕ must be an isomorphism and thus an automorphism.

\Leftarrow : Let H be a core of G . A homomorphism from G to H is an endomorphism, and thus an automorphism. In particular $G = H$, and thus G is a core. □

(*) if $\omega(G) = \chi(G) = k$, then K_k is the core of G .

Vertex transitive graphs

Definition. \triangle A graph G is **vertex transitive** if for any two vertices v and w of G , there is an automorphism that maps v to w .

Example: All cycles and cliques are vertex transitive.

Definition. \triangle The independence ratio of a graph G is $i(G) = \frac{\alpha(G)}{|V(G)|}$.

Lemma (No homomorphism lemma). Let G and H be two graphs such that H is vertex transitive and $G \rightarrow H$, then $i(G) \geq i(H)$.

Proof. (*) Let $S(H)$ be the set of stable sets of H of size $\alpha(H)$. Since H is vertex transitive, we have $\alpha(H)|S(H)| = m|H|$ (where m is the number of independent sets in $S(H)$ containing a specific vertex v). Let ϕ be a homomorphism from G to H . Then for all $I \in S(H)$, we have $|\phi^{-1}(I)| \leq \alpha(G)$ (vertices mapped to an independent set must themselves be independent). Therefore:

$$\sum_{I \in S(H)} |\phi^{-1}(I)| \leq \alpha(G)|S(H)|$$

But $\sum_{I \in S(H)} |\phi^{-1}(I)| = m|G|$ (each $u \in V(G)$ contributes m times to $\sum_{I \in S(H)} |\phi^{-1}(I)|$ since $\phi(u)$ belongs to m members of $S(H)$).

Therefore $i(G) = \frac{\alpha(G)}{|G|} \geq \frac{m}{|S(H)|} = \frac{\alpha(H)}{|H|} = i(H)$. □

Example: \triangle The odd cycles: $C_{2k+1} \rightarrow C_{2k'+1} \Rightarrow \frac{k}{2k+1} \geq \frac{k'}{2k'+1} \Rightarrow k \geq k'$. Therefore by the previous lemma, an odd cycle does not have a homomorphism to a longer odd cycle.

Complexity

\triangle For a fixed graph H , the problem of deciding if a graph G admits a homomorphism to H is called *H-colouring*. It is a generalisation of the k -colouring problems (which corresponds to the case when H is the complete graph K_k).

\triangle The H -colouring problem is in P if H is bipartite, and NP -hard otherwise.

(*) For the other direction: given a fixed graph H , the problem of deciding if a graph G is such that $H \rightarrow G$ is trivially polynomial (just check every set of size at most $|H|$ of $V(G)$). If we consider a whole family \mathcal{H} and try to decide if there exists a graph $H \in \mathcal{H}$ such that $H \rightarrow G$, then the problem is easy if every graph in \mathcal{H} is equivalent (by \leftrightarrow) to a graph of bounded treewidth (with the same bound for the whole \mathcal{H}).