

Hardness reductions

In this lesson there is not much new to know by heart, the aim is to show you methods to do hardness reductions.

⚠ You must be able to reproduce these methods to do other hardness reductions.

Recalls to the complexity of colouring on graphs

See Lesson 6 on colouring algorithms for more details.

We saw a reduction showing that 3-colouring is NP-hard. That reduction implies, under ETH (exponential time hypothesis), that there is no sub-exponential algorithm to solve 3-colouring on general graphs. This is due to the fact that this reduction preserves some specific conditions on the sizes of instances and parameters, that will not be presented in this course (but feel free to read more about this if you are interested). However, we also saw that there is a $O(2^{\sqrt{n}})$ algorithm to solve 3-colouring on planar graphs, although 3-colouring is also NP-hard on planar graphs. This is due to the fact that the reduction from 3-SAT to 3-colouring on planar graphs does not respect the same restrictions as the one to 3-colouring on general graphs.

Independent Set

INDEPENDENT SET: input: a graph G and an integer k .

output: does the graph G admit an Independent Set of size at most k .

Theorem. ⚠ INDEPENDENT SET is NP-hard on general graphs.

Since the problem is NP-hard on general graphs, one can look at smaller classes of graphs. For instance, for a fixed graph H , we can look at H -free graphs. (Recall: ⚠ H -free graphs are graphs that do not have a copy of H as an induced subgraph.)

However:

Theorem. ⚠ If H has a cycle, then INDEPENDENT SET is NP-hard on H -free graphs.

Proof. We do a reduction from INDEPENDENT SET on general graphs. Let $I = (G, k)$ be an instance of INDEPENDENT SET on general graphs. Let us build the graph G' from G by subdividing every edge twice. Let m be the number of edges of G .

Consider an Independent Set S of G' . If there is an edge $uv \in E(G)$ such that both u and v are in S . Since S is an Independent Set in G' , we know that the two vertices that were added on the edge uv cannot be in S . We can remove either u or v from S , say v , and replace it by one of the vertices added on uv . By doing this repeatedly, we can obtain an Independent Set S' of G' with the same size as S , and such that S' does not contain two adjacent vertices of G . Now we can assume for every edge uv of G , one of the two vertices added to uv is in S' , otherwise we can add them to S' . Then if S had $k + m$ vertices, the vertices of $S' \cap V(G)$ form an Independent Set of size at least k in G .

Reciprocally, if G has an Independent Set of size k , we can add one of the vertices added to each edge. This leads to an Independent Set of G' of size $k + m$.

Therefore G has an Independent Set of size k if and only if G' has an Independent Set of size $k + m$. But any cycle of G' is at least three times as long as a smallest cycle in G . Let g be the size of a cycle in H . By applying the previous construction repeatedly, one may ensure that we obtain an instance I' equivalent to I where the smallest cycle in the graph has size more than g . Hence the graph cannot contain H as an induced subgraph, and I' is an instance of INDEPENDENT SET on H -free graphs. \square

Conditional complexity

In this section, we will consider problems where we do not actually know if the answer is always true on the considered classes of graphs. However, we can still prove that, under the condition that the answer is not always true (that is, there exists a counter-example), the problem is *NP*-hard. This method can also be applied for problems where we know actual counter-examples.

We know that triangle-free planar graphs are 3-colorable. One may wonder, however, if forbidding other cycles can also force graphs to be 3-colorable. The following was open for 30 years:

(*) Is every $\{C_4, C_5\}$ -free planar graph 3-colorable? The answer is no (a counter-example was found recently).

However, we have the following:

Theorem. (*) Every $\{C_4, C_5, C_6, C_7\}$ -free planar graph is 3-colourable.

For $i \geq 4$, let \mathcal{C}_i be the class of $\{C_4, \dots, C_i\}$ -free planar graphs. We will prove the following:

Theorem. (*) For all $i \geq 4$, either every graph in \mathcal{C}_i is 3-colourable, or it is *NP*-hard to decide if a graph in \mathcal{C}_i is 3-colourable.

Note that, for $i = 6$, it is still not known whether the problem is trivial or *NP*-hard, but we know that it is either trivial or *NP*-hard.

Proof. \triangleleft The method must be known.

Assume that for a given $i \geq 4$, not every graph in \mathcal{C}_i is 3-colourable. Then there exists a graph in \mathcal{C}_i that is not 3-colourable. Consider such a graph H with minimum number of edges. That is, H is a minimum counter-example to the property that we want to prove.

Let $uv \in E(H)$, and $G' = G \setminus \{uv\}$ be the graph H with the edge uv removed. We have the following properties on H' :

1. Since H' is a subgraph of H , H' is in \mathcal{C}_i , and in particular has no cycle of length from 4 to i .
2. Since H' has fewer edges than H , that is a smallest counter-example, H' has a 3-colouring h .
3. For every 3-colouring of H' , the vertices u and v have the same colour. If not, the 3-colouring of H' would also be proper in H , and H is a counter-example and thus does not admit a proper 3-colouring.

Let us now do a *NP*-hardness reduction from 3-colouring on planar graphs. Consider an instance G of 3-colouring on planar graphs. We will do the following construction on G :

- Take a vertex w of G with a neighbour w' .
- Replace w with a copy of H' , where u has all of the neighbours of w except w' , and v is adjacent to w' .

Let G' be the resulting graph. Let us show that G' has a 3-colouring if and only if G has a 3-colouring.

Let c be a 3-colouring of G . Consider the 3-colouring h of H , that exists by property 2. Note that by property 3, $h(u) = h(v)$. Up to renaming colours in h , we can also assume that $h(u) = c(w)$. Now let $c'(x) = c(x)$ if $x \in G$, and $c'(x) = h(x)$ if $x \in H'$. It is easy to check that c' is a proper 3-colouring of G' .

Let c' be a 3-colouring of G' . Note that the restriction of c' to H' is a proper 3-colouring of H' , and thus by property 3, $c'(u) = c'(v)$. Consider the restriction c of c' to G , where we add that $c(w) = c'(v)$. It is easy to check that c is a proper 3-colouring of G .

Note now that G' is planar, since G and H' are planar, and we branched both graphs on two vertices in the same face of H' and on a vertex of G . Moreover, since $uv \notin E(H')$, u and v have distance at least two in H' . Therefore every cycle of G passing through the edge ww' is increased by at least 2 in G' . Applying this construction repeatedly on every edge ww' of G , we can make sure that every cycle that uses edges of G have length more than i . Therefore the only small cycle that remain in the graph are cycles of H' . By property 1, we know that those small cycles do not have length between 4 and i .

Therefore we built an instance of 3-colouring equivalent to G that is planar and has no cycle with length between 4 and i . So the reduction is done, and 3-colouring is hard on \mathcal{C}_i . \square

Note that this kind of proof can be applied in many settings. If the property that we want to prove is not as symmetrical as the one considered there, for instance if we are trying to prove some partition property, we may have some complications. Namely, there may be several possibilities for the colourings/partitions of the graph H' . In that case, we must consider the different possibilities and do reductions for each of them (possibly nearly the same, possibly different).

Polynomial time lower bounds due to ETH

(*)For some problems, for instance k -Independent Set, we know that the problem is probably not FPT parametrised by k . We say that such instances are $W[1]$ -hard. If ETH is true, then the classes FPT and $W[1]$ -hard are disjoint. From this, and using some specific reductions that we will not see in this course, one can actually prove polynomial lower bounds on the complexity of k -Independent Set. Namely, k -Independent Set cannot be solved in time $n^{o(k)}$.

To prove that, the idea is to take some small (sublinear) function of n as a value of k , and use the $W[1]$ -hard result that would imply (under ETH) that the problem cannot be solved in time $f(k)n^{O(1)}$.