Advanced Graph Theory

(4TIN 917U)

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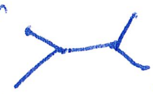
Lecture 2: The probabilistic method

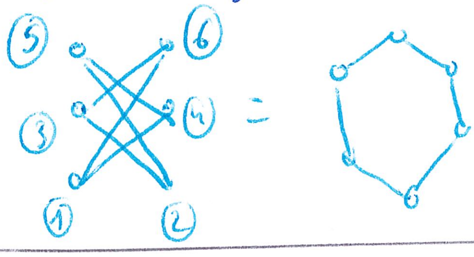
Remember: there is a mentoring program, and a weekly graph seminar (Fridays 2pm).

Flashback to the previous lecture

(a) What is $\alpha($  $)$? $\to 4$ $\quad \chi($  $)$? $\to 2$ $\quad \Delta($  $)$? $\to 3$

$\omega($  $)$? $\to 2$

(b) Example of a graph where some greedy colouring algorithm yields a 3-colouring (and not a 2-colouring)?



(c) Example of a graph G with $\alpha(G) + \omega(G) < |V(G)|$?

(d) Example of a graph G with $\alpha(G) + \omega(G) \leq 2\sqrt{|V(G)|}$?

Theorem (Ramsey 1930)

For any (non-empty) graph G, we have $\alpha(G) + \omega(G) \geq \lfloor \log_2(|V(G)|) \rfloor$

Proof. By induction on $|V(G)|$.

True for $|V(G)| = 1$.

Let $u \in V(G)$. Let $G_u = G[N(u)]$ and $G' = G \setminus (N(u) \cup \{u\})$.

Note that
$$\begin{cases} \alpha(G) + \omega(G) \geq \alpha(G_u) + (\omega(G_u) + 1) \\ \alpha(G) + \omega(G) \geq (\alpha(G') + 1) + \omega(G') \end{cases}$$

If $|V(G')|$ or $|V(G_u)|$ has size more than half $|V(G)|$, the induction holds. Hence $|V(G')| = |V(G_u)| = \frac{|V(G)| - 1}{2}$. Then $\lfloor \log_2(|V(G)| - 1) \rfloor = \lfloor \log_2(|V(G)|) \rfloor$. □

Question: Can we guarantee $\alpha(G)+\omega(G) \geq \lfloor |V(G)|^{\varepsilon} \rfloor$ for any graph G and some $\varepsilon > 0$?

$\hookrightarrow$ Probabilistic Method

⚠ In the model $G_{n,p}$, we build a graph on n vertices by adding every possible edge with probability p, independently.

⚠ What is the probability that $G_{n,1/2}$ contains a clique of size k?

⚠ A subset S of k vertices induce a clique with probability...

$$\frac{1}{2}^{\binom{k}{2}} \sim 2^{-\frac{k^2}{2}}$$

How many subsets of k vertices? $\binom{n}{k} \leq \left(\frac{n}{3k}\right)^k$

Probability is at most $2^{-\frac{k^2}{2}} \times \left(\frac{n}{3k}\right)^k$

If $k = 3\log n$, very unlikely

⚠ $\alpha(G_{n,1/2}) + \omega(G_{n,1/2}) \leq 6 \log n$

(With high probability the $G_{n,1/2}$ process results in a graph such that...)

⚠ Consequently, the bound in Ramsey's theorem is tight up to a constant factor.

① What is the probability that $G_{n,1/2}$ contains a long induced path on $3\log n$ vertices?

② What is (with high probability) the diameter of $G_{n,1/2}$?

⚠ The answer to whether $\alpha(G)+\omega(G) \geq |V(G)|^{\varepsilon}$ for some $\varepsilon>0$ and $\forall$ G is $\boxed{\text{NO}}$. Constructing $\neq$ Showing existence

* Conjecture (Erdös-Hajnal '89)

For every graph H, $\exists$ $\varepsilon_H > 0$ such that $\forall$ G,

- either G contains an induced copy of H (i.e. $G[X] \cong H$ for some $X \subseteq V(G)$)
- or $\alpha(G)+\omega(G) \geq |V(G)|^{\varepsilon_H}$

Very widely open. Breakthrough this year: true for $H =$ .

* Still open for  ...

* $G_{n,p}$: Erdös-Renyi model for random graphs.

Many other models (fixed number of edges, regular graphs etc).

⚠ Useful to prove that there are graphs w/ arbitrarily high chromatic number and arbitrarily large girth (Lecture 1: girth 4).

* Consider $G_{n,\frac{1}{n^{1-\frac{1}{2\ell}}}}$, where ℓ is the desired lower bound on the girth. Informally, there are probably some cycles of length $<\ell$, but "few" of them (whp). There is no "large" stable set (whp). Delete all small cycles $\rightsquigarrow$ still no "large" stable set (in proportion to the total number of vertices). In the resulting graph (which isn't a random graph in the Erdös-Renyi model) there is no cycle of length less than ℓ, and if n is large enough, the chromatic number is high.

Exercise ① What are $\alpha\left(G_{n,\frac{1}{\sqrt{n}}}\right)$ and $\omega\left(G_{n,\frac{1}{\sqrt{n}}}\right)$, roughly?

② Prove Erdös-Hajnal's conjecture for H being a clique
(Hint: by induction on $|V(H)|$)

* Conjecture: $\forall G$, $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$

* Theorem (Molloy '17): $\forall$ triangle-free G,
$$\chi(G) \leq (1+o(1)) \cdot \frac{\Delta(G)}{\ln(\Delta(G))}$$

Proof by the probabilistic method (using something magical called Lovasz Local Lemma).

Harder to construct corresponding solution

⚠ Theorem (Turan '41)

$\forall$ graph G on n vertices (with $n \geq 1$),
$$|E(G)| \leq \left(1 - \frac{1}{\omega(G)}\right) \times \frac{n^2}{2}$$

* Proof
- Choose a random ordering on the vertices.
- Build a clique C_0 greedily according to the ordering.
- $\forall u \in V(G)$, the probability that $u \in C_0$ is $\frac{1}{n-d(u)}$
- Expectation of the size of C_0 is

$$\sum_u \frac{1}{n-d(u)} \quad \text{(by linearity of expectation)}$$

However, we have $\omega(G) \geq \mathbb{E}(C_0) = \sum_{u \in V(G)} \frac{1}{n-d(u)}$

$$n^2 = \left(\sum_{u \in V(G)} \frac{1}{\sqrt{n-d(u)}} \times \sqrt{n-d(u)}\right)^2$$
$$\leq \left(\sum_{u \in V(G)} \frac{1}{n-d(u)}\right)\left(\sum_{u \in V(G)} (n-d(u))\right) \leq \omega(G) \cdot (n^2 - |E(G)|)$$

* **Theorem (46):**

$\forall$ H with at least one edge, $\forall$ graph G on n vertices,

- either G contains H as a subgraph, or
- $|E(G)| \leq \left(1 - \frac{1}{\chi(H)-1} + o(1)\right) \cdot \frac{n^2}{2}$

Proof by Szemeredi's regularity lemma

* Extremely hard to get a good bound when H is bipartite ($\chi(H)=2$)

Exercise: How many (labelled) graphs on n vertices?

How many (labelled) bipartite graphs on n vertices?

Options for the next lecture:

(A) Edge-colouring (by Jonathan)

(B) Graph Minors (by François)

(C) Edge decomposition and sparsity (by François)