

Edge Coloring

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1 Definition

Definition 1 (*k*-edge-coloring).

Let $G = (V, E)$ be a graph, a *k*-edge-coloring φ of G is a mapping:

$$\begin{aligned} E(G) &\rightarrow \{1, \dots, k\} \\ e &\mapsto \varphi(e) \end{aligned}$$

Such that if e and e' share a common vertex, then $\varphi(e) \neq \varphi(e')$.

Note that this definition is also valid for multigraphs, but we won't consider them during this lesson.

If such a coloring exists, G is called *k*-colorable, and the *chromatic index*, denoted by $\chi'(G)$ is the minimum number of colors to color the edges G .

A *k*-edge-coloring can equivalently be defined as a vertex coloring of $L(G)$, the *line graph* of G .

Definition 2 (Line graph).

Let G be a graph, the *line graph* of G , denoted by $L(G)$ is the graph with vertex set $E(G)$, and such that two vertices are adjacent in $L(G)$ if the two edges they represent share a common vertex in G .

It is clear that a graph G is *k*-edge-colorable if and only if $L(G)$ is *k*-vertex-colorable.

The *class of line graphs*, is the class of graphs H s.t. there exists a graph G s.t. $H = L(G)$. However Beineke [Bei70] (*) proved that this class can be characterized by a set of 9 forbidden subgraphs (see Figure 1), ie a graph is in the class if it does not contain as an **induced subgraph** one of these graphs.

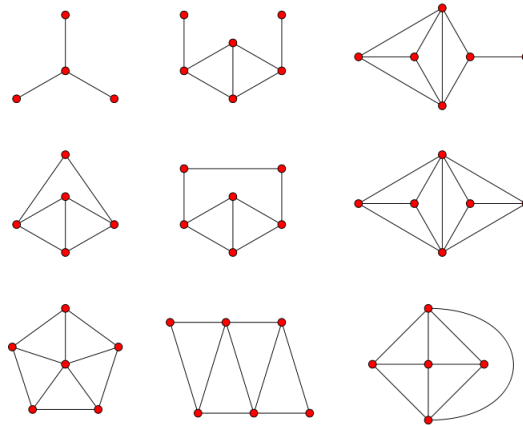


Figure 1: The 9 forbidden subgraphs characterizing the class of line graphs

(*)**Example 3** (example of line graphs).

Consider a chessboard, a Rook's graph is a graph that represents all legal moves of the rook chess piece on this chessboard (see Figure 2). This graph is the line graph of the complete bipartite graph $K_{n,n}$ (where n is the width and the height of the chessboard)

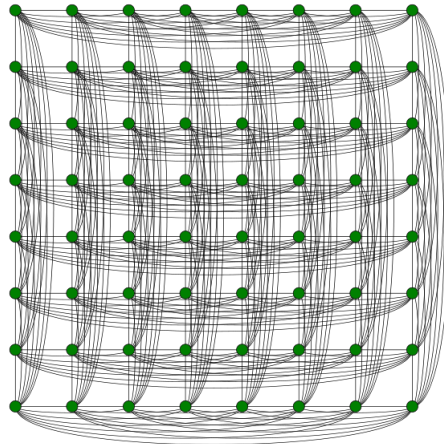


Figure 2: A 8×8 Rook's graph

2 Bounding of the chromatic index

2.1 Simple bounds

As all the edges sharing a common vertex should receive a different color, it is easy to see that we'll need at least the maximum degree of the graph colors to properly color the edges of G , so we have that:

$$\triangle \chi'(G) \geq \Delta(G),$$

where $\Delta(G)$ denotes the maximum degree of G . Note that there exist graphs for which $\Delta(G)$ colors suffice, for instance: a star (ie a set of vertices with one vertex connected to every other vertex).

To find an upper bound for $\chi'(G)$, it suffices to consider the bound given by a greedy vertex-coloring algorithm of the line graph of G . Recall that for any graph G , we have that $\chi(G) \leq \Delta(G) + 1$ (where $\chi(G)$ is the minimum number of colors to properly color the vertices of G). So if we apply this bound to $L(G)$, we have that $\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1$.

As we have that $\Delta(L(G)) = 2 \times \Delta(G) - 2$, we finally have that:

$$\triangle \chi'(G) \leq 2\Delta(G) - 1$$

2.2 Better upper bound: the case of bipartite graphs

The $2\Delta(G) - 1$ upper bound is clearly not optimal, and König proved in 1931 [Kon31] (in Hungarian :-() that $\Delta(G)$ colors always suffice if G is bipartite.

\triangle Theorem 1 (König's theorem).

Let G be a bipartite graph, then $\chi'(G) = \Delta(G)$.

The proof of this theorem relies on a standard tool in graph coloring : *Kempe changes*. Kempe changes allows one to transform a proper coloring into another proper coloring: it can be used to reduce the number of colors needed to color a graph. Kempe changes were invented by Alfred Kempe in the 19th century to prove the 4-color theorem (see Section 4). Kempe's proof was false, but Kempe changes turned out to be a fruitful tool to study graph coloring.

Definition 4 (Kempe chain for edge-coloring).

Let G be a graph and φ a k -edge coloring of G . We consider the subgraph of G , $K(a, b)$ induced by the color classes a and b (ie we only keep the edges colored a or b in G). Let C be a connected component of $K(a, b)$, C is called a Kempe chain. A Kempe change consists in switching the colors of the edges in C (ie, all the edges that were colored a are now colored b and vice-versa). After the switch, we obtain a new coloring φ' which is also a proper edge-coloring of G .

Note that this definition can also work for vertex coloring. However, when considering edge-coloring, the Kempe chains can only be one of these three different graphs:

- An even cycle:

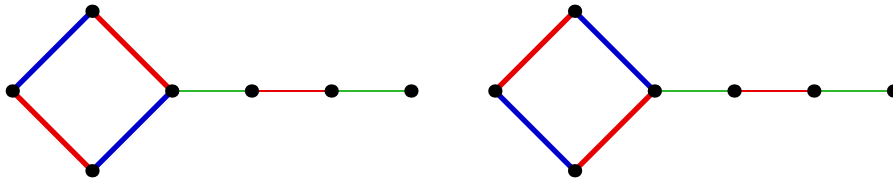


Figure 3: A red-blue Kempe switch on an even cycle

- A path:

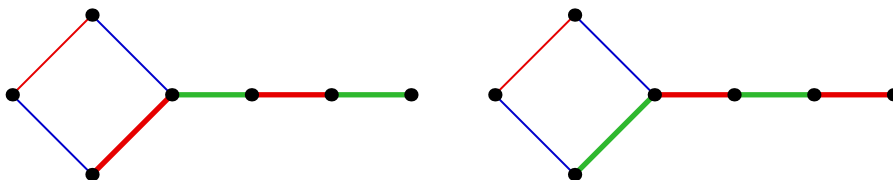


Figure 4: A red-green Kempe switch on a path

- A single edge:

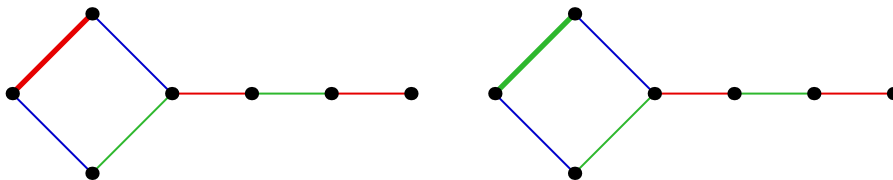


Figure 5: A red-green Kempe switch on a single edge

Note that in the case where the Kempe chain is a single edge, it is possible to reduce the number of colors. The C_4 is colored with 3 colors before the Kempe change, and with only 2 after switching the single-edge Kempe chain.

In the case of vertex-coloring, the Kempe chains can be any arbitrary bipartite graphs, and hence can be less easy to manipulate.

We are now ready to prove König's theorem.

Proof. The proof is a proof by minimum counter example. Assume that there exist bipartite graphs which need more than Δ colors to properly color their edges. Among them, we choose G such that G has the minimum number of edges, and let $e = uv$ be an edge of G . Let $G' = G \setminus \{e\}$, ie the graph G where we delete the edge e . The graph G' is also a bipartite graph, and has less edges than G , so there exists a Δ -coloring φ' of G' . We are now going to extend the coloring φ' of G' to a coloring φ of G . As the edge e does not have any color yet, the vertices u and v are only incident with at most $\Delta(G) - 1$ different colors, so there exist colors in $\{1, \Delta\}$ which are missing at u and v , we respectively denote them m_u and m_v . We now have to distinguish two cases:

- If $m_u \cap m_v \neq \emptyset$, then there exists a color c which is missing at u and at v , and so it suffices to color e with this color to obtain a proper coloring φ of G

- If $m_u \cap m_v = \emptyset$, then there is no available color for e , and we need to transform the coloring φ' using kempe changes to free a color for e . Let $a \in m_u$, and $b \in m_v$, we consider the component C_u of $K(a, b)$ containing the vertex u .

We now prove that C_u cannot contain the vertex v . Assume it is the case, then C_u is a path connecting u to v , but as u is missing the color a and v is missing the color b , it means that the length of C_u is even. However, in G u and v are connected by the edge e , so we have that $C = C_u \cup \{e\}$ is a cycle in G . But the length of C_u is even, so the length of C is odd, which is a contradiction as G is bipartite (recall that in a bipartite graph, there are only even cycles, see Figure 6).

So finally we have that C_u does not contain v , and it suffices to switch C_u to obtain a coloring where u and v are missing the color b , we can hence extend φ' to G by giving to e the color b .

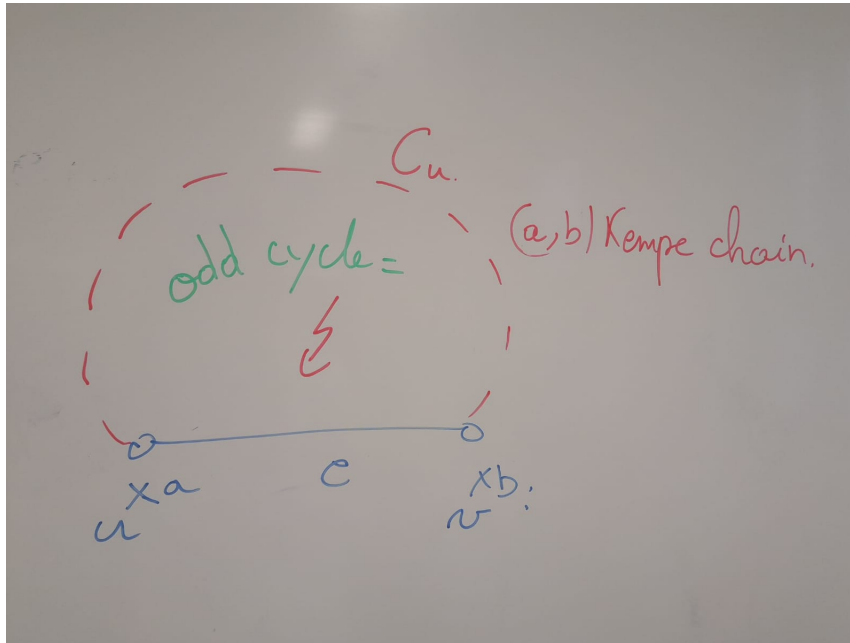


Figure 6: The Kempe chain C_u forms an odd cycle with e

□

2.3 Better bound: the general case

So bipartite graphs only need Δ colors to properly color their edges. Moreover, Vizing proved in 1964 [Viz64] (in Russian :-), that with only one more color than the lower bound, it is always possible to properly color the edges of a graph.

⚠ **Theorem 2** (Vizing's theorem).

Let G be a graph, then $\chi'(G) \leq \Delta(G) + 1$.

The proof of this theorem also relies heavily on the use of Kempe changes, but is more involved than the proof of König's theorem. To prove the theorem, Vizing invented an extension of the Kempe changes : the *Vizing's fans*.

(*) **Definition 5** (Vizing's fans).

Let G be a graph, and φ a k -coloring of G , with $k > \Delta$. Each vertex u of G is missing at least one color. Let u be a vertex of G , and $\{u_1, \dots, u_\Delta$ the neighbors of u . A fan around u starting at uu_i is a maximal sequence of edge $(uu_i = uu_{i_0}, uu_{i_1}, \dots, uu_{i_k})$ s.t. for any j in $\{1, \dots, k\}$, $\varphi(uu_{i_j}) \in m_{u_{i_{j-1}}}$ where m_w is the set of missing colors at the vertex w .

(*) *Proof.*

Similarly to the previous proof, we consider a minimum counter example G , and a $(\Delta + 1)$ -coloring φ' of $G' =$

$G \setminus \{e\}$ where $e = uv$ is an edge of G . We will extend φ' to a coloring φ of G . As φ' is a $(\Delta + 1)$ -coloring of G' , each vertex w of G miss at least one color, and we denote by m_w this set of missing color at w .

If there exists a colors $c \in m_u \cap m_v$, then this color is available for e , and we can extend φ' to a proper $(\Delta + 1)$ -coloring of G .

So there is common missing color at u and v . We will now use the Vizing fans around u starting at uv to free a color for e . So we consider $X = \{uv = uu_0, \dots, uu_l\}$ a fan around u starting at uv . As X is maximal, we only have two cases to consider.

- if $m_u \cap m_{u_l} \neq \emptyset$:

Then we can apply a single edge Kempe change on the edge uu_l and by definition of X obtain a coloring where $m_{u_{l-1}} \cap m_u \neq \emptyset$. So we can continue to apply single edge Kempe changes on the edge uu_i until we reach a coloring where $m_u \cap m_v = m_{u_0} \neq \emptyset$, and so we have a free color for e .

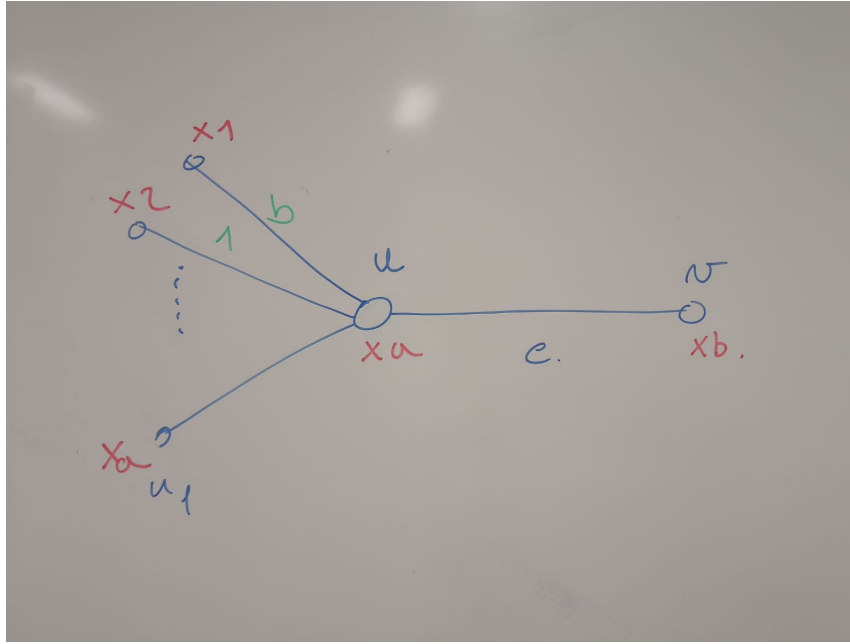


Figure 7: A missing color at u is also missing at u_l

- there exists $j \in \{0 \dots, l-1\}$ s.t. $m_{u_j} \cap m_{u_l} \neq \emptyset$:

In this case, let $a \in m_u$, $b \in m_{u_l} \cap m_{u_j}$, and let C_l be the component of $K(a, b)$ containing u . The component C_l is a path, and it contains u , so it cannot contain both u_l and u_j . The two cases are symmetric, so wlog we can consider that C_l does not contain u_l . So we can apply a Kempe change on the component C'_l of $K(a, b)$ containing u_l without changing the missing colors at the vertices u_i , nor the colors of the edges of X . After the Kempe change, we have a coloring where $m_{u_l} \cap m_u \neq \emptyset$, so there is a fan $X' = \{uv = uu_0, \dots, uu_{l'}\}$ around u starting at uv s.t. $m_u \cap m_{u_l} \neq \emptyset$, which correspond to the previous case: so by a sequence of single-edge Kempe changes we can obtain a coloring where $m_u \cap m_v \neq \emptyset$, and color e with one color of this set.

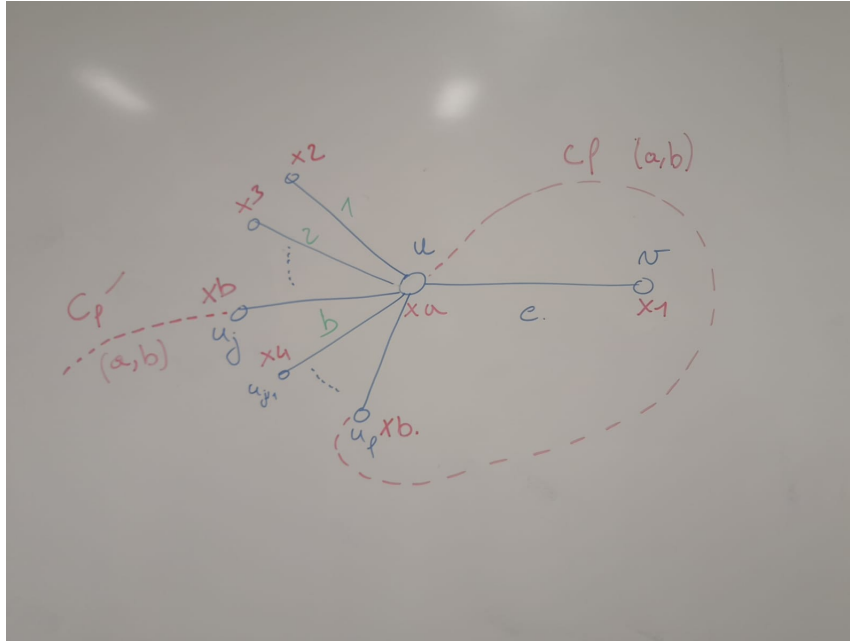


Figure 8: A missing color at u_l is also missing at u_j , and the Kempe chain C_l does not contain u_j .

□

Note that the original statement of Vizing's theorem is on multigraph, he proves that $\chi'(G) \leq \Delta(G) + \mu(G)$ where $\mu(G)$ is the edge-multiplicity of G (ie the maximum number of edges between two vertices). Moreover, the proof of the theorem does not only give a bound on the number of colors, but yields an algorithm to transform any edge-coloring of G into a $(\Delta + 1)$ -edge-coloring of G .

3 More on edge coloring

3.1 List edge-coloring

Similarly to vertex-coloring, we can generalize of edge-coloring to list-edge-coloring using lists of available colors for the edges. Each edge e is assigned a list $L(e)$ of possible colors for this edge (each edge have its own set of possible colors). We say that a graph is L -list-edge-colorable if the edges of G are colorable with the colors in their list. The question is then: what is the minimum k s.t. G is L -list-edge-colorable with L being a list assignment with all lists of size at least k . This minimum size of the lists is called (*) *edge choosability number*, and is denoted by $ch'(G)$.

Using a greedy algorithm, it is easy to see that for any graph G , (*) $ch'(G) \leq 2\chi'(G)$; however it is conjectured that for any graph G , $ch'(G) = \chi'(G)$. This conjecture is still widely open, Galvin proved in 1995 [Gal95] that it is true for complete bipartite graphs.

(*)**Theorem 1** (Galvin).

Let $n \in \mathbb{N}$, $ch'(K_{n,n}) = \chi'(K_{n,n}) = n$.

Note that this has an equivalent reformulation in term of Rook's graphs.

3.2 Complexity of computing χ'

Despite the fact that by Vizing theorem, $\chi'(G)$ can only take two values, Δ or $(\Delta + 1)$, it is (*) NP-complete to decide whether $\chi'(G) = \Delta$ or $\chi'(G) = \Delta + 1$ [Hol81], even for very simple classes of graphs (for instance triangle-free graphs [Kor97]).

(*)**Example 6** (subdividing an edge).

For instance, K_4 the complete on 4 vertices is $\Delta(K_4) = 3$ -edge colorable, but if we subdivide one of its edge (ie we replace an edge by a path of length 2), the graph is no more Δ -colorable.

(*)**Example 7** (complete graphs).

The case of the complete graph K_n only depends on the parity of n . If n is even, then K_n is $(n - 1) = \Delta$ -colorable, and if n is odd, K_n is $n = (\Delta + 1)$ -colorable.

(*)**Example 8** (planar graphs).

The case of planar graphs is almost settled, Vizing proved in 1965 [Viz65] that all planar graphs with $\Delta \geq 8$ are Δ -colorable (the proof uses the **discharging method**), and it was later proved ([Zha00], [SZ01]), that planar graphs with maximum degree 7 are also Δ -colorable. On the other hands, for graph with maximum degree less than 5, we have examples of planar graphs which need $(\Delta + 1)$ colors; the only remaining case are planar graphs with maximum degree 6.

3.3 Reconfiguration questions

As we have seen in the proof of Vizing's theorem, the key to reduce the number of colors needed to color the edge of a graph is to reconfigure an existing coloring in order to free an available color for each edge. However, the question of reconfiguring edge-coloring is of its own interest. In his paper of 1965, (*) Vizing asks the question whether it is always possible to reach an optimal coloring starting from a coloring using more colors, only using Kempe changes.

This question can be generalized in the following way: starting from a k -edge-coloring (with $k > \Delta$), can we reach any χ' -edge-coloring of G using only Kempe changes. It is conjectured that it is true for all graphs, but it was only proved for specific class of graphs, namely bipartite graphs [Asr09], graphs of maximum degree 4 [AC16], and triangle-free graphs [BDK⁺21]; all these proofs rely heavily on the use of Vizing fans.

4 Edge-coloring as a tool for vertex coloring

Edge-coloring can also be a tool to study problems on vertex coloring. The famous 4-color theorem states that if G is a planar graph, then 4 colors suffice to properly color the vertices of G . This question was asked by Guthrie in the 19th century, and it took more than a century to get a formal and complete proof of the theorem. The proof of the theorem is based on the discharging method, but is so long and involved that it can only be checked using a computer (see [AH76], or [RSST97] for a more recent proof with the actual code in C to check it). However, the first step of the proof is to reduce the problem of 4-coloring the vertices of a planar graph G to a problem of 3-edge-coloring or the dual of G .

Definition 9 (Dual graph).

Let $G = (V, E, F)$ be a planar graph (with F being the set of faces of G), the dual of G , denoted by G^* is the graph whose vertex set is F , and two vertices of G^* are adjacent if the faces they represent share an edge in G ; G is called the primal of G^* .

Note that taking the dual graph of the dual graph gives back the primal graph G , and that each edge in the primal graph correspond to an edge in the dual graph.

To prove the 4-color theorem, it suffices to prove it on triangulation (graphs whose faces are only triangles). It is easy to see that if G is a planar graph, we can add edges to G to build a graph G' where all the faces are triangles: if we can find a 4-coloring φ of G' , then φ is also a coloring of G .

Let G be a triangulation, we will prove that G has a 4-vertex coloring if and only if G^* has a 3-edge coloring.

(*)**Theorem 2.** Let G be a triangulation, then G is 4-vertex-colorable if and only if G^* is 3-edge colorable.

(*) *Proof (sketch).* Assume that we have a 4-vertex-coloring φ of G using the colors $\{1, 2, 3, 4\}$, then we can partition the edges of G according to the colors of their endpoints. Let A be the set of edges whose endpoints are colored with $\{1, 2\}$ or $\{3, 4\}$, B the set of edges whose endpoints are colored with $\{1, 4\}$ or $\{2, 3\}$, and C the set of edges whose endpoints are colored with $\{1, 3\}$, or $\{2, 4\}$. We now define a coloring φ' of G^* by coloring the edges with colors $\{a, b, 0\}$ such that the edges respectively in the sets A , B , or C are colored a , b or 0 (see Figure 9). As G is a triangulation, all the vertices in G' have degree 3. Moreover, it is easy to see that for a triangle f in G , each edge of f is in a different set (otherwise we would have that the three vertices of a triangle are colored with 4 colors), so for each vertex f^* of G^* , each edge incident with f^* is colored with a different color, and hence φ' is a proper edge-coloring of G .

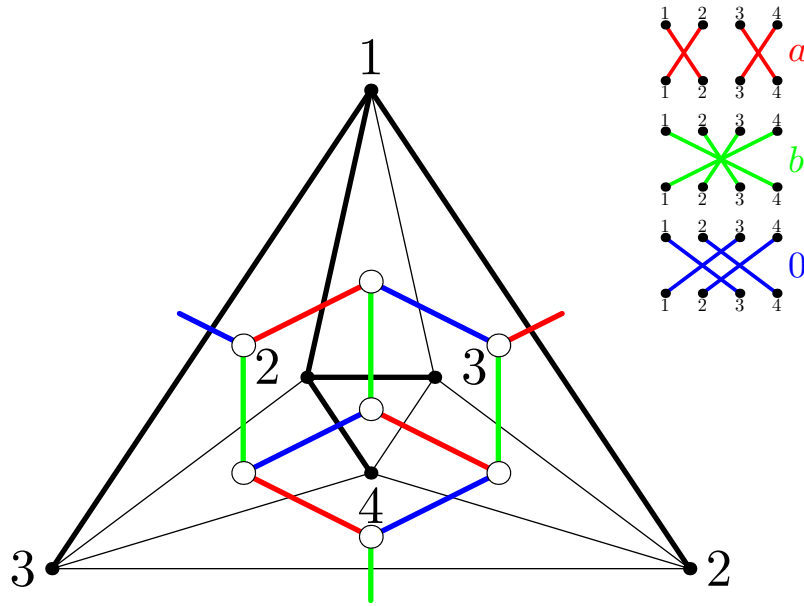


Figure 9: The edge-coloring of the dual corresponding to a vertex coloring of the primal graph (the three outgoing edges are connected to the vertex representing the outerface of the primal graph which is not drawn for legibility). Note that the bold edges form a spanning tree of G that can be used to build the vertex-coloring starting from the edge coloring of G^*

Conversely, assume that we have a 3-edge coloring φ' of G^* using the colors $\{a, b, 0\}$. To find a 4-coloring of G , it suffices to consider a spanning tree T of G , and choose a root r of this tree. We color r with color 1, and color each vertex u of G according to the color of its parent v in the tree following the same rules defined in the first part of the proof:

- if $\varphi'(uv) = a$
 - if $\varphi(v) = 1$, then $\varphi(u) = 2$
 - if $\varphi(v) = 2$, then $\varphi(u) = 1$
 - if $\varphi(v) = 3$, then $\varphi(u) = 4$
 - if $\varphi(v) = 4$, then $\varphi(u) = 3$
- if $\varphi'(uv) = b$
 - if $\varphi(v) = 1$, then $\varphi(u) = 4$
 - if $\varphi(v) = 4$, then $\varphi(u) = 1$
 - if $\varphi(v) = 2$, then $\varphi(u) = 3$
 - if $\varphi(v) = 3$, then $\varphi(u) = 2$
- if $\varphi'(uv) = 0$
 - if $\varphi(v) = 1$, then $\varphi(u) = 3$
 - if $\varphi(v) = 3$, then $\varphi(u) = 1$
 - if $\varphi(v) = 2$, then $\varphi(u) = 4$
 - if $\varphi(v) = 4$, then $\varphi(u) = 2$

We leave as an exercise for the reader the proof of the coloring φ being a proper vertex coloring of G . □

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