

Lecture 4: Graph Minors (Marthe BONAMY)




Flashback to the previous lecture.

(a) What is the minimum number of vertices in a graph G that is not $\Delta(G)$ -edge-colourable?

(b) Can we decide in polynomial time whether the degree n p vertices admits a $(p-1)$ -edge-colouring?

⚠ A graph H is an induced subgraph of G if $\exists X \subseteq V(G)$ such that $G[X]$ is isomorphic to H .





Alternatively, H is obtained from G by a series of vertex deletions.
(Delete successively all vertices in $V(G) \setminus X$).

Example:  is an induced subgraph of  but  is not.

⚠ A graph H is a subgraph of G if H can be obtained from G through a series of vertex and edge deletions.

Example:  and  are both subgraphs of .  is not.

⚠ A graph H is a minor of G if H can be obtained from G through a series of vertex and edge deletions and edge contractions.

Example: , ,  are all minors of .  is not.

⚠ $\{G \mid C_3 \text{ is not a minor of } G\}$? All forests.

* $\{G \mid K_n \text{ is not a minor of } G\}$? Series-parallel graphs

* $\{G \mid K_4 \text{ and } K_{2,3} \text{ are not minors of } G\}$? Outerplanar graphs

Δ {Planar graphs} is a class that is closed under taking minors. 2/

Δ {Planar graphs} = {G | K_5 and $K_{3,3}$ are not minors of G}

* Theorem (Robertson, Seymour 83-04)

If graph class \mathcal{G} that is closed under taking minors, there is a finite list of graphs H_1, \dots, H_p (for some $p \in \mathbb{N}$) such that $\mathcal{G} = \{G \mid H_1, \dots, H_p \text{ are not minors of } G\}$.

Δ There is an $O_H(n^2)$ algorithm to decide whether a graph G on n vertices admits H as a minor, where H is fixed.

* Consequently, there is (theoretically) a $O(n^2)$ algorithm to decide whether a graph G on n vertices belongs to a fixed class \mathcal{G} closed under taking minors.

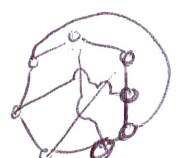
* What about graphs that can be drawn in the plane without crossing edges? Also closed under taking minors.

* At least 17500 minimal forbidden minors (full list unknown)

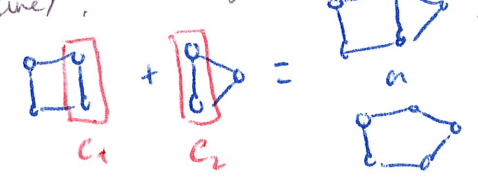
* What about {G | K_5 is not a minor of G}?

* Theorem (Wagner '37)

Every K_5 -minor-free graph is either a planar graph, or ~~the~~ obtained from two smaller K_5 -minor-free graphs by a clique-sum.



Let H_1, H_2 be two K_5 -minor-free graphs. Let C_1 be a clique in H_1 and C_2 a clique in H_2 , with $|C_1| = |C_2|$ (≤ 4 for sure). We merge H_1 and H_2 as $C_1 = C_2$, and possibly delete some edges. For example



What about graphs with no K_p -minors, for $p \geq 6$?

Two ingredients:

Connected to Treewidth
Optim

- Clique-sums \rightarrow
- ⚠ Graphs embedded in a given bounded genus surface.

Example of such surfaces: Torus, Möbius ribbon, Klein bottle, pretzels... Very roughly, genus = number of holes in the surface.

Not as neat a characterization as for $p \leq 5$ but still nice!

⚠ Cherem (Kostochka, Thomas '82)

$\forall \epsilon$, Every graph either admits K_ϵ as a minor or contains a vertex of degree $\leq c \cdot \epsilon \sqrt{\log \epsilon}$

⚠ A graph G is h -degenerate if $\forall X \subseteq V(G)$, $G[X]$ contains a vertex of degree $\leq h$.

Example: A forest is 1-degenerate.

A planar graph is 5-degenerate.

Best possible because of random graphs!

K_ϵ -minor-free graphs are $O(\epsilon \sqrt{\log \epsilon})$ -degenerate.

⚠ Any h -degenerate graph is $(h+1)$ -colorable.

Question: Example of a h -degenerate graph that is not h -colorable?

* Conjecture (Hadwiger '43)

$\forall \epsilon$, every graph G ~~either admits K_ϵ as a minor or~~ ~~subgraph~~ with $\chi(G) \geq \epsilon$ admits K_ϵ as a minor.

For $\epsilon = 5$, equivalent to the 4 color theorem (strong form!)

$\epsilon = 6$: common man's trick den. Réémmeur (Koply) Option 2: FPT and discharging
 $\epsilon = 7$: Osvet!

* Méthode probabiliste pour l'instant inutile dans l'attaque de la conjecture.

K_C -minor-free graphs generalize planar graphs (and a lot more) and behave well with respect to many properties


For example, they admit balanced separators of size $O(\sqrt{|V(G)|})$.
(Also related to treewidth)

Back to other substructures

What if we forbid H as an (induced) subgraph?

If H contains a cycle, H -free graphs may have arbitrarily high chromatic number.

* If H is a forest, $\{G \mid H \text{ is not a subgraph of } G\}$ has bounded chromatic number.

(Example: if H is a star. )

* $\{G \mid H \text{ is not an induced subgraph of } G\}$ may contain arbitrarily large digraphs (hence arbitrarily high χ)

* Conjecture (Gyarfas '80).

\forall forest F , $\exists f$,

$\forall G$ with no induced F as a subgraph, $\chi(G) \leq f(|V(G)|)$.

Known for $F =$ path or star.

Proof of him.

Option 3: χ -bounded graph class

