

# Advanced Graph Theory (4TING17U)

Narthe BONARTY

## Lectures 5 and 8: Discharging and Fixed Parameter Tractability (Or how to get rid of useless stuff)

Flashback to the previous lecture: complexity to recognize if a graph is planar? Poly-time since characterized by finitely many forbidden minors.

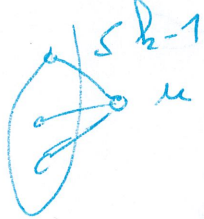
Example of infinitely many pairwise incomparable (for the induced subgraph relation) trees?



Consider a problem (say  $k$ -coloring) and an instance of it.

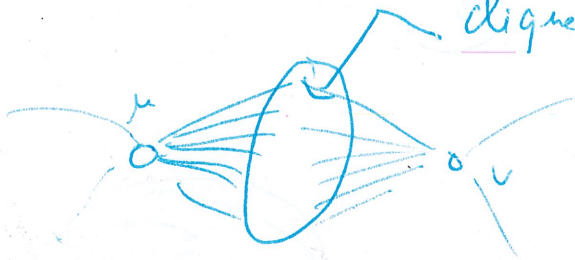
Goal: Simplify the instance as much as possible.

Examples:



Vertices of degree  $\leq k-1$  can be ignored (deleted).

dique on  $k-1$  vertices.



In any solution,  $u$  and  $v$  have the same color as merge  $u$  and  $v$  into a single vertex.

Things to worry about:

- Did I lose information? (ex: there is a vertex adjacent to all the other ones, and the graph is planar)
- Did I modify the problem? (ex:  $k$ -coloring  $\rightarrow$   $(k-1)$ -coloring)
- Did I leave the graph class? (ex: the graph is planar and I merged vertices)

New goal: Prove that all planar graphs are 5-colorable.

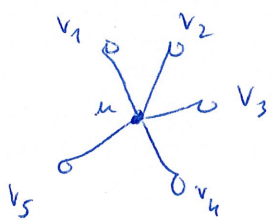
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Problem: 5-coloring Instance: A planar graph  $G$ .

① Vertices of degree  $\leq 4$  are irrelevant for the problem (can be deleted)

② Vertices of degree 5 are also irrelevant.

Proof 1:



If  $G$  contains  $u$  with  $d(u) = 5$   
assume  $G \setminus u$  admits a 5-coloring  $\alpha$   
Consider a planar embedding of  $G$  and let  $v_1, \dots, v_5$  be the neighbors of  $u$  in clockwise order.  
If  $\{v_1, v_2, v_3, v_4, v_5\}$  use  $\leq 4$  colors in  $\alpha$ ,

then  $\alpha$  extends to  $G$ . Otherwise, assume without loss of generality

that  $\alpha(v_i) = i \quad v_i \in \{1, 2, 3, 4, 5\}$ . Consider a ~~Kempe~~ Kempe chain (1,3) on vertex  $v_1$  (Remember, this is a maximal component colored 1 and 3 in  $\alpha$  that contains  $v_1$ ).

If it does not contain  $v_3$ , we can swap the two colors and go back to the previous case. So there is a path colored 1-3 between  $v_1$  and  $v_3$  (possibly  $v_1 v_3$  is an edge). By the same analysis, there is a path (possibly just an edge) colored 2-4 between  $v_2$  and  $v_4$ . The two paths are vertex-disjoint since they use disjoint colors. Since this contradicts the planarity of the embedding of  $G$ .

Proof 2: If  $v_i v_j$  is an edge  $\forall i \neq j$ , then  $G$  contains a  $K_5$  on 5 vertices, a contradiction with planarity. Therefore,  $\exists i \neq j$  s.t.  $v_i v_j \notin E(G)$ . Consider the graph  $G'$  obtained from  $G$  by contracting the edges  $v_i u$  and  $u v_j$ .  $G'$  is planar since planarity is closed under taking minors. Assume  $G'$  admits a 5-coloring  $\alpha$ . We derive a 5-coloring  $\alpha'$  of  $G \setminus u$  that uses  $\leq 4$  colors on  $\{v_1, \dots, v_5\}$ , hence the conclusion.




# Advanced Graph Theory (4TIN917U)

Marthe BONAMY

## Lectures 5 and 8: Discharging and Fixed Parameter Tractability

(Or how to get rid of useless stuff)

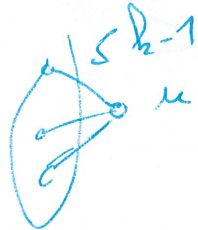
Flashback to the previous lecture: complexity to recognize if a graph is planar? Poly-time since characterized by finitely many forbidden minors.

Example of infinitely many pairwise incomparable (for the induced subgraph relation) trees? 

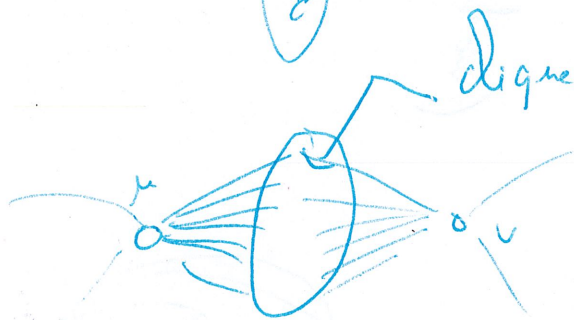
Consider a problem (say  $k$ -coloring) and an instance of it.

Goal: Simplify the instance as much as possible.

Examples:



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degree  $\leq k-1$  vertices.

In any solution,  $u$  and  $v$  have the same colour  $\rightarrow$  merge  $u$  and  $v$  into a single vertex.

Things to worry about:

- Did I lose information? (ex: there is a vertex adjacent to all the other ones, and the graph is planar)
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It suffices now to argue that all planar graphs whose vertices all have degree  $\geq 6$  are 5-colourable.

We use

$\Delta$  Euler's formula: for every planar embedding  $(V, E, F)$  of a planar graph  $G$ , we have  $|V| - |E| + |F| > 0$

\*(in fact =  $1 + \#$  connected components)

The traditional proof is by easy induction, here we show an informal proof following a discharging argument.

$\Delta$  Discharging = Double counting

"Silly" example: Gauss' formula for  $\sum_{i=1}^n i$

Weight assignment  $w: i \mapsto i$

Discharging rule:  $\forall i < \frac{n+1}{2}$  gives all its weight to the element  $n+1-i$

Let  $w'$  be the final weight (after application of the discharging rule)

$\forall i < \frac{n+1}{2}, w'(i) = 0$

$\forall i > \frac{n+1}{2}, w'(i) = n+1$

If  $n$  is odd,  $w'(\frac{n+1}{2}) = w(\frac{n+1}{2}) = \frac{n+1}{2}$

$\sum_{i=1}^n i = \sum_{i=1}^n w(i) = \sum_{i=1}^n w'(i) = \frac{n(n+1)}{2}$

1	2	3	4	5	6	7	$= w$
0	0	0	4	8	8	8	$= w'$

Easier to count



# Euler's formula

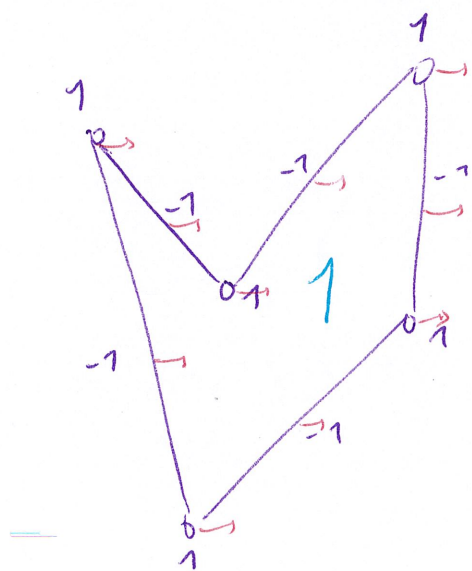
$$|V| - |E| + |F| = \sum_{v \in V} 1 + \sum_{e \in E} (-1) + \sum_{f \in F} 1$$

Weight assignment:  $\forall v \in V, w(v) = 1$

$\forall e \in E, w(e) = -1$

$\forall f \in F, w(f) = 1$

Consider ~~an orientation~~ an axis for which no edge is perfectly horizontal.



Axis

Right

Discharging rule: Every vertex and every edge gives all its weight to the face on its right.

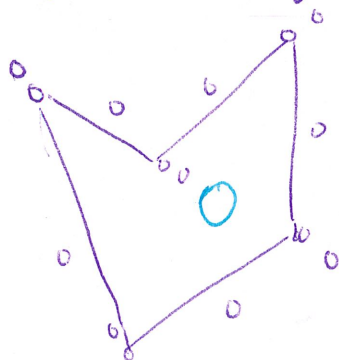
$w'$  the resulting weight assignment.

$\forall v \in V, w'(v) = 0$

$\forall e \in E, w'(e) = 0$

$\forall f \in F$  if  $f$  is an internal face,  $w'(f) = 0$

$w'(\text{external face}) = 1 + \# \text{ connected components}$



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We have

$$|V| - |E| + |F| = \sum_{x \in V \cup E \cup F} w(x) = \sum_{x \in V \cup E \cup F} w'(x) = w'(\text{external face}) = 1 + \# \text{ connected components} > 0$$

Note that

$$\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6)$$

$$= 2|E| - 6|V| + 4|E| - 6|F| < 0$$

Note that as soon as  $G$  contains at least two edges, we have  $d(f) \geq 3 \quad \forall f \in F$ .

$$\text{Therefore } \sum_{f \in F} (2d(f) - 6) \geq 0.$$

If every vertex has degree at least 6,

$$\text{then } \sum_{v \in V} (d(v) - 6) \geq 0$$

A contradiction since  $\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) < 0$ .

Consequently, every planar graph  $G$  with  $V(G) \neq \emptyset$  contains a vertex of degree  $\leq 5$ .

This implies that all planar graphs are 5-colorable, as desired.

Can be seen as a proof by induction.

Usually:  $H_0$  holds. If  $H_n$  holds then  $H_{n+1}$  holds.

Here: the property holds for the empty graph. If  $G$  contains a vertex  $u$  of degree  $\leq 5$ , then the property for  $G - u$  implies the property for  $G$ .

⊕ Work to guarantee all graphs are built



A finer discharging argument.

Goal: Every planar graph  $G$  with  $V(G) \neq \emptyset$

contains

(a) a vertex of degree  $\leq 4$ , or

(b) a vertex of degree 5 adjacent to a vertex of degree  $\leq 6$ .

Proof:  $(V, E, F)$  a planar embedding of  $G$ .

$$w(v) := d(v) - 6 \quad \forall v \in V(G)$$

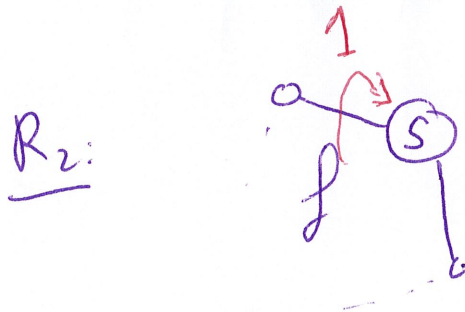
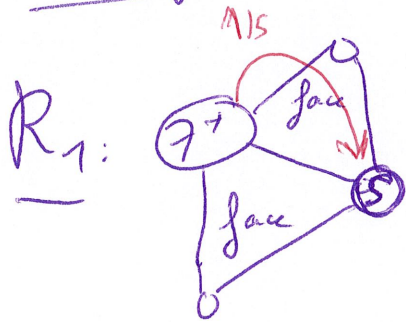
$$w(f) := 2d(f) - 6 \quad \forall f \in F$$

When  $G$  contains at least two edges, so  $\forall f \in F, 2d(f) - 6 \geq 0$ .

$$\sum_{x \in V \cup F} w(x) < 0 \quad \text{by Euler's formula.}$$

$x \in V \cup F$

Discharging rules:



Let  $w'$  be the resulting weight assignment.

Assume for a contradiction that  $V(G) \neq \emptyset$  and  $G$  contains neither (a) nor (b).

Let  $v \in V, d(v) \geq 5$ .

If  $d(v) = 5, w(v) = -1$ . If  $v$  is incident to a face  $f$  of degree  $\geq 4$ , by  $R_2$   $v$  receives 1 at least, so  $w'(v) \geq 0$ . If  $v$  is only incident to triangles, then all its neighbors have degree  $\geq 7$  so  $R_1$  applies 5 times, hence  $v$  receives  $5 \times \frac{1}{5} = 1$  so  $w'(v) \geq 0$ .

If  $d(v) = 6$ , no rule applies, so  $w'(v) = w(v) = 0$ .

If  $d(v) \geq 7$  by  $R_1$  and since there is no edge  $(S) - (S)$  by assumption on  $G$ ,  $R_1$  is applied at most  $\lfloor \frac{d(v)}{2} \rfloor$  times on  $v$ .

Therefore  $w'(v) \geq w(v) - \frac{1}{5} \times \lfloor \frac{d(v)}{2} \rfloor \geq \frac{9}{10}d(v) - 6$

Note that  $w'(v) \geq 0$  when  $d(v) \geq 7$ .

Therefore,  $\forall v \in V, w'(v) \geq 0$

Let  $f \in E$ .

If  $d(f) = 3$  no rule applies so  $w'(f) = w(f) = 0$

If  $d(f) \geq 4$ , since there is no edge  $(S) - (S)$ ,  $R_2$  is applied at most  $\frac{d(f)}{2}$  times.

Hence  $w'(f) \geq w(f) - 1 \times \frac{d(f)}{2} \geq \frac{3d(f)}{2} - 6$

Note that  $w'(f) \geq 0$  when  $d(f) \geq 4$ .

Consequently,  $\sum_{z \in E \cup F} w'(z) \geq 0$

Since  $\sum_{z \in E \cup F} w(z) = \sum_{z \in E \cup F} w(z) < 0$ , we reach a contradiction.

D.



# Fixed Parameter Tractability.

Problem is FPT wrt parameter  $k$  if  $\exists$  algorithm in time  $f(k) \cdot \text{poly}(\text{instance})$  for some computable function  $f$ .


Equivalently, the problem admits a kernel of size  $g(k)$  that can be computed in  $\text{poly}(\text{instance})$ .

Example: Vertex Cover parametrized by the size of the solution?

Given  $(G, k)$ , does there exist  $S \subseteq V(G)$  s.t.

$$\begin{cases} |S| \leq k \\ \forall uv \in E(G), \{u, v\} \cap S \neq \emptyset \end{cases}$$

Reduction 1: If  $G$  contains a vertex  $u$  of degree 0, return  $(G \setminus \{u\}, k)$ .

Reduction 2:  If  $G$  contains a vertex  $v$  of degree  $\geq k+1$ , return  $(G \setminus \{v\}, k-1)$ .  
 "v has to be in the solution if any"