

Graph Partition/Decomposition

Definition. \triangle Let $\mathcal{C}_1, \dots, \mathcal{C}_k$ be k classes of graphs. A $\mathcal{C}_1, \dots, \mathcal{C}_k$ -partition of a graph $G = (V, E)$ is a partition of V into k sets V_1, \dots, V_k such that for all $i \in \{1, \dots, k\}$, $G[V_i] \in \mathcal{C}_i$.

(*)Let:

- I denote the class of graphs with no edges (or empty graphs).
- F denote the class of graphs with no cycles (or forests).
- Δ_d denote the set of graphs with maximum degree at most d .
- F_d denote the set of forests with maximum degree at most d .
- δ_d denote the set of graphs with degeneracy at most d .

Planar graphs

Exemples: The four colour theorem: Every planar graph admits an (I, I, I, I) -partition.

Theorem. (*) Every planar graph admits a (δ_1, δ_2) -partition.

Theorem. (*) Every planar graph admits a (δ_0, δ_3) -partition.

Remark: $\delta_0 = I$; $\delta_1 = F$.

Ideas of proofs: To prove these result, one needs to apply induction on stronger assumptions, that is that some prepartition/precolouring of some of the vertices, generally vertices on the outerface, can be extended to the whole graph. Note that as usual, we can instead look at smallest counter-examples to this stronger assumptions and prove that they do not exist.

\triangle More generally, results of this kind are either proven by the discharging method, or using the ideas above.

We will see an example of a proof of this kind later in this course.

Definition. \triangle A k -acyclic colouring is a proper k -colouring of a graph such that the subgraph induced by any pair of colours is a forest.

In other words, it must be a k -colouring such that there is no cycle with vertices of exactly two colours.

Theorem. \triangle Every planar graph admits a 5-acyclic colouring.

The proof is very complicated, with a lot of configurations, here is why 5 colours are necessary:

Proof. We give a counter-example in Figure 1.

Consider the colours 0, 1, 2, and 3. In this graph, suppose w_0 and w_1 are coloured the same, say with colour 0 without loss of generality. The vertex u_0 needs to have a distinct colour, say 1, so that the colouring is proper. Similarly, v_0 must have a fresh colour, say 2 without loss of generality. But now u_1 cannot have colour 0 or 2 because the colouring must be proper, and it cannot have colour 1, otherwise $u_0 w_0 u_1 w_1$ would be a cycle with colours 0 and 1. So u_1 must have colour 3. Now v_1 cannot have colours 0, 1, or 3 for the colouring to be proper, so it has colour 2, but now $v_0 w_0 v_1 w_1$ is a cycle with two colours, 0 and 2, a contradiction.

Therefore w_0 and w_1 are not coloured the same. But up to renaming, the pairs of vertices (u_0, u_1) , (v_0, v_1) , and (w_0, w_1) behave the same. We say that there is an isomorphism that maps (u_0, u_1) to (v_0, v_1) , and one that maps (u_0, u_1) to (w_0, w_1) (see the course on homomorphisms). Thus u_0 and u_1 are not coloured the same, and v_0 and v_1 are not coloured the same either. Hence, together with the condition of the coloring being proper, we know that no two vertices can have the same colour. But there are six vertices and only four colours, a contradiction. \square

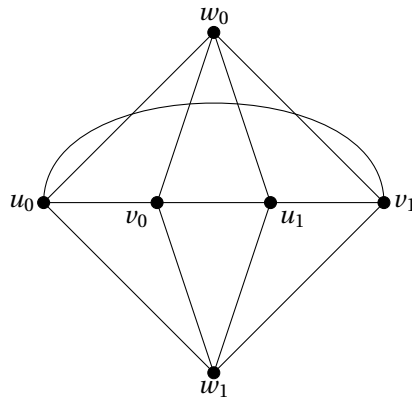


Figure 1: A counter-example to the fact that every planar graph has a 4-acyclic colouring.

Corollary. *Every planar graph admits an (I, F, F) -partition.*

(*) This is the best possible for this kind of partitions. For four sets, the best possible is of course the four colour theorem. There exists a planar graph that admits no (I, I, F) -partition. We do not give it in this course.

(*) That graph does not admit an (F, F) -partition either, since every forest admits an (I, I) -partition (every forest is bipartite). We will give another explanation of why there exists a planar graph with no (F, F) -partition: Consider a triangulation G (a planar graph where every face is a triangle). The dual \hat{G} of G is a 3-regular planar graph. Reciprocally, any 3-connected 3-regular planar graph has a dual that is a triangulation. Let us now look at an (F, F) -partition of G . In the dual \hat{G} , this partition corresponds to a partition of the faces into two sets. However, there is no cycle in the same partition of G , and thus, since G is a triangulation, the boundary separating these two sets of faces has to be a single cycle. This cycle goes through every vertex of \hat{G} (otherwise there would be a cycle in a set of the partition around the isolated vertices), and thus is a hamiltonian cycle. However, it is known that there exists a (bipartite) 3-connected 3-regular planar graph that admits no hamiltonian cycle (any counter-example to Tutte's conjecture will do. The interested reader can find more on Tutte's conjecture easily on the internet). The dual of such a graph has no partition into two forests.

Triangle-free planar graphs

△ Recall that triangle-free planar graphs are 3-colorable.

(*) They admit an (F, F) -partition:

Proof. Assume the opposite, and consider a smallest counter-example G (according to the number of vertices). Since this counter-example is a triangle-free planar graph, it admits a vertex v with degree at most 3. $G \setminus \{v\}$ is smaller than G , so it cannot be a counter-example. Therefore there exists a partition of $G \setminus \{v\}$ into two forests. Now since v has degree at most 3, one of those forests contains at most one neighbour of v . Adding v to this forest cannot create a cycle. Therefore we have an (F, F) -partition of G , a contradiction. □

(*) Do they admit an (F, I) -partition? This is an open problem. (known for planar graphs with girth at least 5).

Recall: △ the **girth** of a graph is the size of a smallest cycle in the graph.

(*) But what if we only look at 3-degenerate (not necessarily planar) graphs? After all, the previous proof works for any 3-degenerate graph. K_4 is a counter-example.

(*) The precise statement is not to know / △ the process of how to find a counter-example is to know. Look at Figure 2 for an illustration of the different gadgets we build.

Ok, but K_4 has a lot of triangles. What about triangle-free 3-degenerate graphs? Let us look for a counter-example. For simplicity, we will here just call I the independent set and F the forest in the partition. We want to build gadgets that force some things. We will keep in mind that our graph needs to be triangle-free and 3-degenerate. First, by considering a cycle, we are sure that at least one vertex will be in I . We of course can not take a triangle. We can take a C_4 , for example (see Figure 2, left). Now we can look for a gadget where, given a vertex in I , we reach a contradiction.

In our second gadget, we have a vertex u that we can assume is in I . We can give it two neighbours, say u_0 and u_1 , that will be in F (recall that I is an independent set). Now if we add two additional common neighbours to u_0 and u_1 , say v and w , at least one of these will need to be in I (see Figure 2, middle).

In our third gadget, that we will branch both on u and v and on u and w in the previous gadget, we can assume that two vertices, x and y , are in I . Recall that these two vertices have a common neighbour in the previous gadget, so we cannot just add an edge between x and y . We can, however, add two neighbours to x and two neighbours to y , and build a 4-cycle on them without making a triangle (see Figure 2, right).

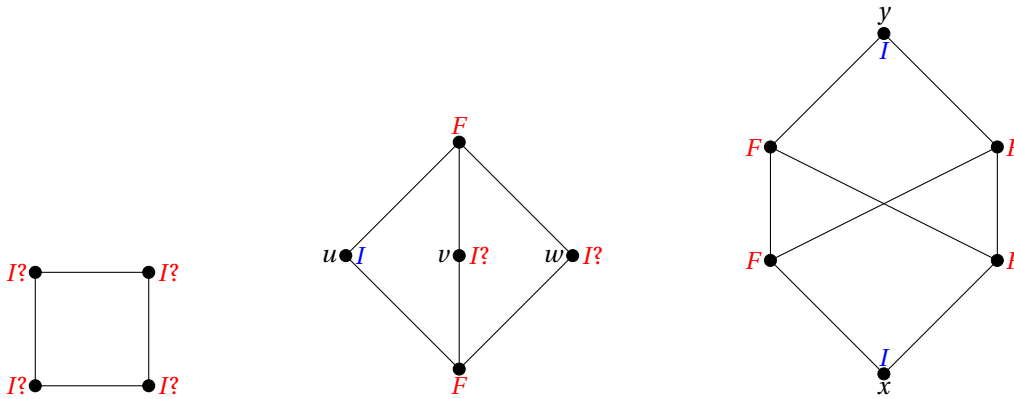


Figure 2: The building of a counter-example to a property, namely that every 3-degenerate triangle-free graph admits an (I, F) -partition. On the left the first gadget, a cycle. We will branch the second gadget (at vertex u) on every vertex. On the second gadget, in the middle, we branch the third gadget between u and v on the one hand, and between u and w on the other hand. The third gadget, on the right, is branched at vertices x and y . The labelings in red correspond to the sets that are forced by the current gadget. The interrogation marks mean that at least one of those vertices must be in the corresponding set. The labelings in blue correspond to the sets forced by the previous gadget (assuming that the red information is followed in the previous gadget).

Now the obtained graph does not admit an (I, F) -partition. It clearly does not contain a triangle (the only potential problem would have been between x and y in the third gadget, and we made sure not to make them adjacent). Now we need to prove that it is 3-degenerate. To do this, we will specify an order for the vertices to be removed such that the vertices have degree at most 3 when removed. The vertices in F in the third gadget are 3-vertices, let us remove them first (in every instance of this gadget). Now the third gadget has completely been removed. Let us remove all of the vertices besides u in each instance of the second gadget (they have degree 2 or 3). Now the graph is a cycle, and we can remove the remaining gadgets.

The graph is therefore a 3-degenerate triangle-free graph with no (I, F) -partition, which is what we wanted.

Improper/defective coloring

Definition. \triangle A (k, d) -improper colouring is a colouring with k colours, such that the graph induced by each colour has degree at most d .

In our setting, a (k, d) -improper colouring is thus a $(\Delta_d, \Delta_d, \dots, \Delta_d)$ -partition, where Δ_d appears k times.

Definition. \triangle An outer-planar graph is a graph that can be drawn in the plane such that every vertex is on the same face (by convention, the outerface).

Definition. (*) A linear forest is a forest with maximum degree 2. (that is a collection of paths).

Theorem. (*) Outerplanar graphs admit a $(2, 2)$ -improper colouring, and even an (F_2, F_2) -partition, but not a $(2, 1)$ -improper colouring.

Theorem. \triangle Planar graphs admit a $(3, 2)$ -improper colouring, and even an (F_2, F_2, F_2) -partition, but not a $(3, 1)$ -improper colouring.

Proof. \triangle Here again, the process is to know. (You need to be able to reproduce it).

We will prove the following stronger statement:

For each near-triangulation (each face except maybe the outerface is a triangle) G , with the outerface $v_1 \dots v_k v_1$ separated into two induced paths $p_1 = v_1 \dots v_r$ and $p_2 = v_{r+1} \dots v_k$, we can partition the set of vertices into three linear forests, the first one with p_1 as a component, and the second one with p_2 as a component.

We consider a smallest counter-example G to that statement. We have two cases to consider.

- There exist $i < j$ such that $i \in \{1, \dots, r\}$ and $j \in \{r+1, \dots, k\}$, with $(i, j) \neq (r, r+1)$ and $(i, j) \neq (1, k)$ such that $v_i v_j \in E(G)$. See Figure 3. Then this edge separates the graph into two smaller subgraphs, G_1 and G_2 , that share only vertices v_i and v_j in common. We can apply induction hypothesis on each subgraph. The first linear forest of G_1 has $v_1 \dots v_i$ as a component, and the first linear forest of G_2 has $v_i \dots v_r$ as a component, we can merge them together into a linear forest with p_1 as a component. Similarly, we can merge the second linear forests of both subgraphs into a linear forest with p_2 as a component. Lastly, we can merge the third linear forests, that have no vertex in common, into a single linear forest.

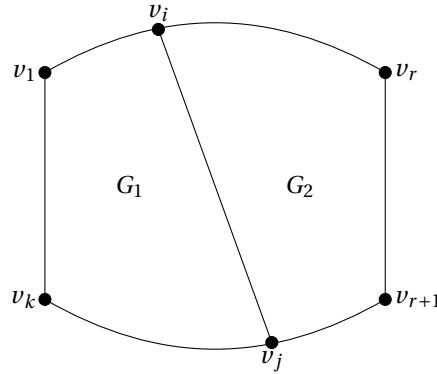


Figure 3: The first case, with an edge $v_i v_j$ between the two paths.

- The previous case does not apply. Then removing the boundary of the outerface does not disconnect the graph. Now since we are in a near-triangulation, there is a vertex a forming a triangle $av_1 v_k$ and a vertex b forming a triangle $bv_r v_{r+1}$. Let us consider a shortest path p_3 connecting a and b without using vertices of the boundary of the outerface. Since we took the shortest path, it is indeed an induced path (it has no chord). Now we can apply induction on the graphs G_1 and G_2 , bounded by $p_1 p_3$ and $p_3 p_2$, respectively.

Let (F_1^1, F_2^1, F_3^1) be the three linear forests, in order, obtained from G_1 , and (F_1^2, F_2^2, F_3^2) be the three linear forests, in order, obtained from G_2 . Now $(F_1^1 \cup F_3^2)$ is a linear forest with p_1 as a component (the other components cannot interfere). $(F_3^1 \cup F_2^2)$ is a linear forest with p_2 as a component (similarly). And F_2^1 and F_1^2 are linear forest that both have p_3 as a component, so they do not interfere, and $(F_2^1 \cup F_1^2)$ is a linear forest.

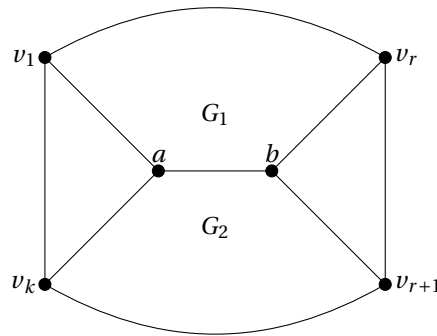


Figure 4: The second case, no edge between the two paths.

Now we have proven the stronger statement. Let us prove that it is indeed stronger than what we wanted to prove. Let G be a planar graph, and G' be a triangulation that is a supergraph of G (just add edges while you can). Now the outerface of G' is a triangle $v_1 v_2 v_3$, you can take $v_1 v_2$ as p_1 and v_3 as p_2 , and apply the statement. Now we have a partition of G' into three linear forests. The same sets induce linear forests in G as well, since a subgraph of a linear forest is a linear forest.

□

Edge decomposition

We did not have time to say much, just the definition:

Definition. \triangle Let $\mathcal{C}_1, \dots, \mathcal{C}_k$ be k classes of graphs. A $\mathcal{C}_1, \dots, \mathcal{C}_k$ -decomposition of a graph $G = (V, E)$ is a partition of E into k sets E_1, \dots, E_k such that for all $i \in \{1, \dots, k\}$, $(V, E_i) \in \mathcal{C}_i$.