Coloring the squares of graphs

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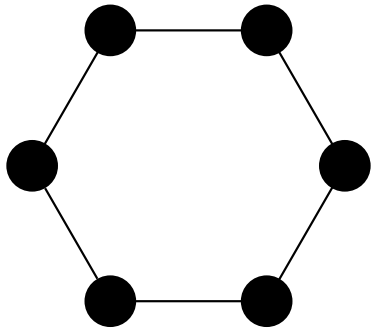
Montpellier

Square coloring



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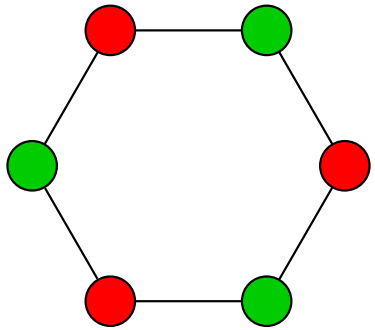
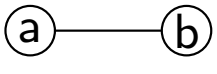


χ: Minimum number of colors to ensure that



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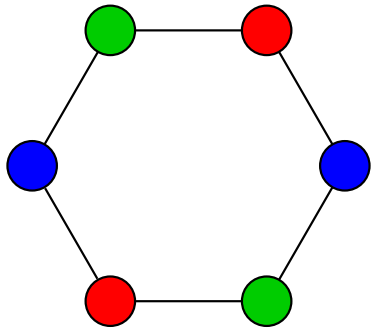

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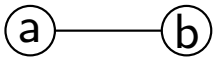

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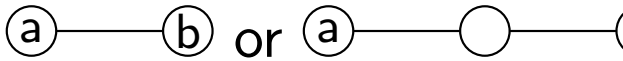

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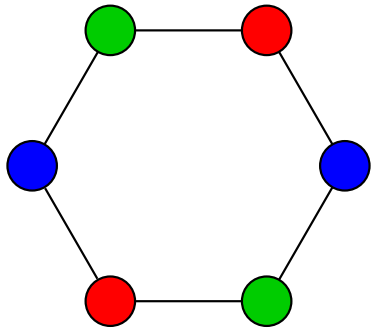

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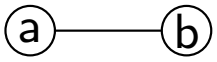

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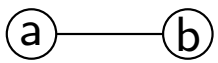

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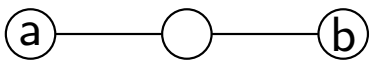

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Δ: Maximum degree of the graph.

$$\Delta^2 + 1 \geq \chi^2 \geq \Delta + 1.$$

Square coloring

A graph is planar if it can be drawn in a plane without crossing edges.

Conjecture (Wegner 1977)

If G is a planar graph, then:

- $\chi^2 \leq 7$ *if* $\Delta = 3$
- $\chi^2 \leq \Delta + 5$ *if* $4 \leq \Delta \leq 7$
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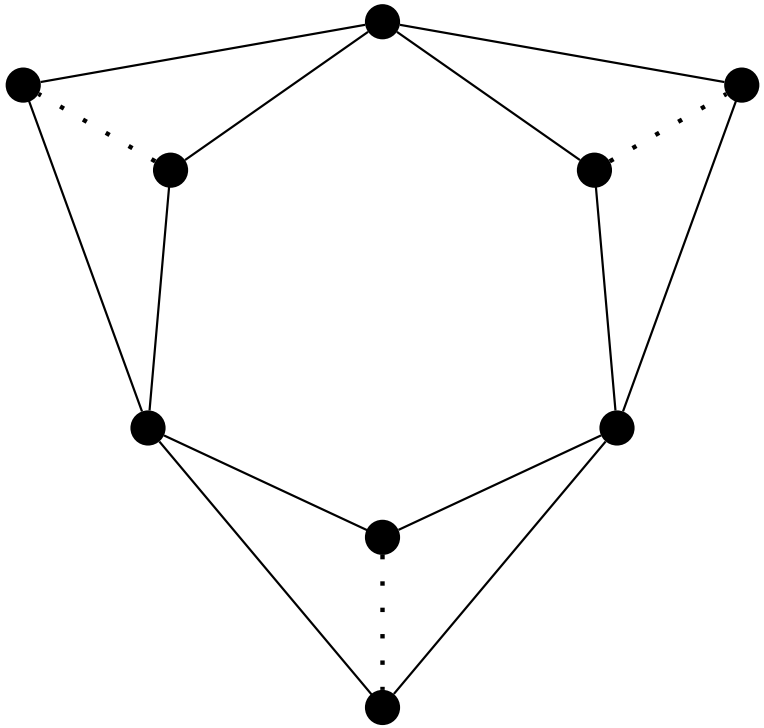

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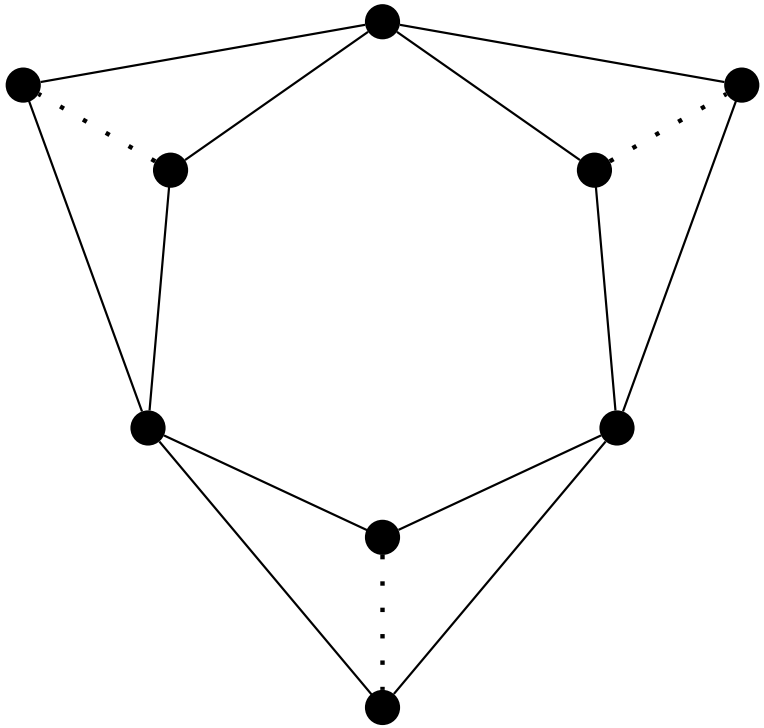

Theorem (Bu, Zhu '11)

Wegner's conjecture is true for planar graphs with $g \geq 6$.

Sparse planar graphs (2)

Question (Wang, Lih '01)

$$g \geq k,\ \Delta \geq d \Rightarrow \chi^2 = \Delta + 1.$$

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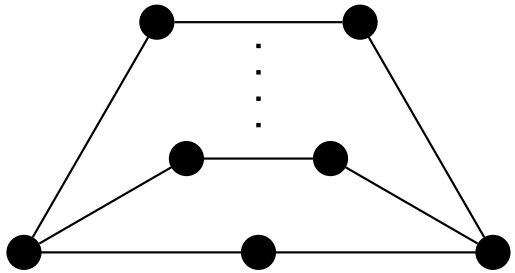
Theorem (Borodin, Glebov, Ivanova, Neustroeva, Tashkinov '04)

False for $k \leq 6$. *True for* $k \geq 7$.



State of the art

Theorem (Borodin, Ivanova, Neustroeva '08 & Ivanova '11)

For any planar graph, $\chi^2 = \Delta + 1$ in each of the following cases :

- $g \geq 24$ *and* $\Delta \geq 3$
- $g \geq 15$ *and* $\Delta \geq 4$
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- $g \geq 10$ *and* $\Delta \geq 6$
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$$\sum_{f \in F} \underbrace{(d(f) - g)}_{\geq 0} + c \cdot \sum_{v \in V} (d(v) - \frac{2g}{g-2}) < 0$$

Another measure of sparseness

ad: average degree $= \frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|}$.

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For any planar graph, $(\text{mad} - 2)(g - 2) < 4$

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$$M \text{ maximal} \mid \forall\, m < M,\ \exists\, d,$$
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First results on non-planar sparse graphs

Theorem (B., Lévêque, Pinlou '11, Cranston, Škrekovski '13)

For any graph, $\chi^2 = \Delta + 1$ in each of the following cases :

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What about $mad < 3$?

First results on non-planar sparse graphs (2)

Theorem (B., Lévêque, Pinlou '13)

For a small enough ϵ, $\mathrm{mad} < 3 - \epsilon$ *and* $\Delta \geq \frac{3}{\epsilon^2} \Rightarrow \chi^2 = \Delta + 1$.

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Theorem (Borodin, Kostochka, Woodall '97)

For any bipartite multigraph G, *if* L *is an edge list assignment such that* $|L(u,v)| \geq d(u), d(v)$ *for every* (u,v), *then* G *is L-edge-choosable.*

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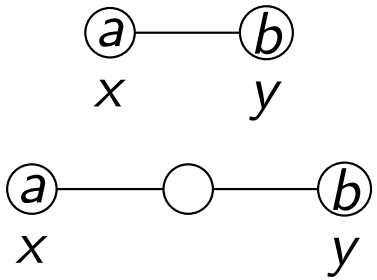
$\Rightarrow$ Global discharging method.

List coloring?

χ^2_ℓ: Minimum size of every $L(v)$ such that



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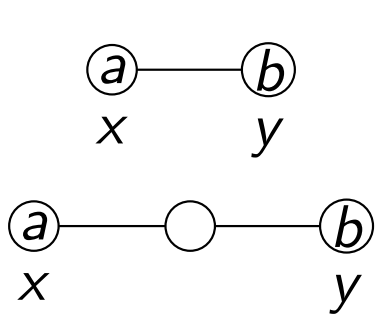

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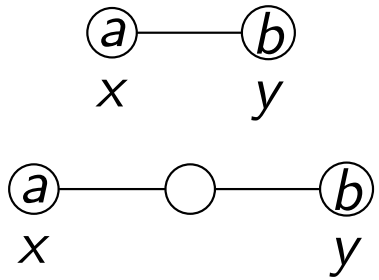

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What changes in the previous theorems?

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Conjecture (Kostochka, Woodall '01)

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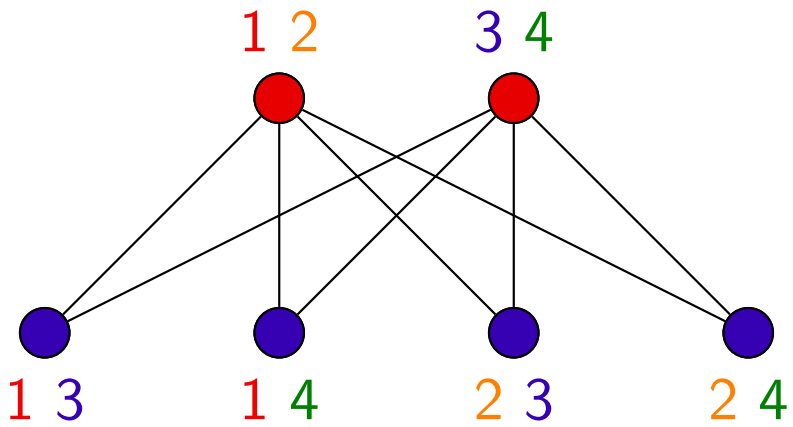

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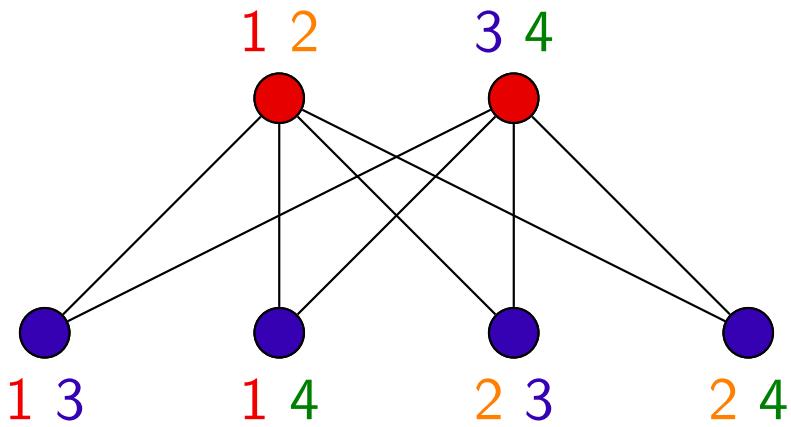

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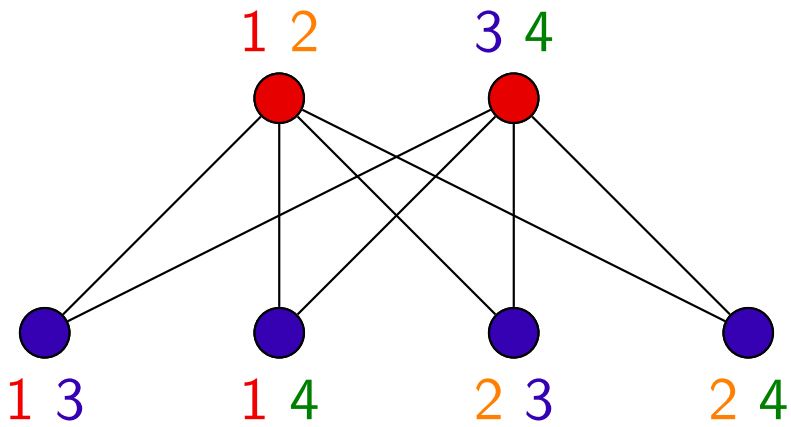

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Theorem (Kim, Kwon, Park '13, Kosar, Petrickova, Reiniger, Yeager '13)

For any k, $\chi^k_\ell - \chi^k$ *can be arbitrarily large.*

Upper bounds

$$\Delta^2 + 1 \geq \chi_\ell^2 \geq \chi^2 \geq \Delta + 1.$$

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Conjecture (Cranston, Kim '08)

If $\Delta \geq 3$, no graph satisfies $\chi_\ell^2 = \Delta^2$.

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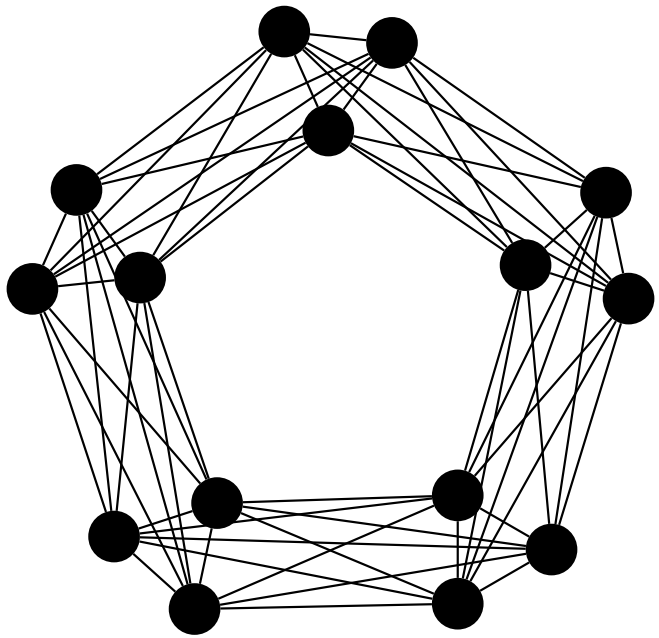

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Conjecture (Miao, Fan '13)

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Conclusion

Conjecture

For any $k \geq 3$, *and* $\Delta \geq 3$, *no graph satisfies*

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For any C *and* $\epsilon > 0$,

$\text{mad} < \frac{4C+2}{C+1} - \epsilon$ *and* $\Delta \geq f(C, \epsilon) \Rightarrow \chi_\ell^2 \leq \Delta + C$.