

CR16: Notes on χ -bounded graph classes*

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1 Ramsey's theorem

Theorem 1 (Ramsey '30). *For every (non-empty) graph G , we have $\alpha(G) + \omega(G) \geq \frac{1}{2} \cdot \log_2 |V(G)|$.*

Proof. By induction on $|V(G)|$. If $|V(G)| = 1$, the conclusion holds. Assume now that $|V(G)| \geq 2$. Let $u \in V(G)$. Let $G_u = G[N(u)]$ and $G' = G \setminus N[u]$. Note that $\alpha(G) + \omega(G) \geq \alpha(G_u) + (1 + \omega(G_u))$ and $\alpha(G) + \omega(G) \geq (1 + \alpha(G')) + \omega(G')$.

Since $|V(G)| = 1 + |V(G_u)| + |V(G')|$, one of $|V(G_u)|$ and $|V(G')|$ has size at least $\frac{|V(G)|}{4}$. We obtain $\alpha(G) + \omega(G) \geq 1 + \frac{1}{2} \log_2(\frac{|V(G)|}{4})$, hence the conclusion. \square

2 No large star

Theorem 2 (Gyárfás '87). *For every t , there is a function f_t such that any $K_{1,t}$ -free graph G satisfies $\chi(G) \leq f_t(\omega(G))$.*

Proof. Let $t \in \mathbb{N}$. For every vertex $u \in V(G)$, the graph $G[N(u)]$ contains no clique on $\omega(G)$ vertices and no stable set on t vertices. By Theorem 1, we derive that $d(u) \leq 2^{2\omega(G)+2t}$. The conclusion follows immediately. \square

3 No long path

Theorem 3 (Gyárfás '87). *For every t , there is a function f_t such that any P_t -free graph G satisfies $\chi(G) \leq f_t(\omega(G))$.*

Proof. We prove by induction on t and $\omega(G)$ that there is a function f_t such that for any connected graph G and any vertex $u \in V(G)$, either $\chi(G) \leq f_t(\omega(G))$ or u is the endpoint of an induced path on t vertices in G . Note that the statement holds trivially when $t = 1$, and we assume from now on $t \geq 2$.

Consider $G \setminus N[u]$, and let C_1, \dots, C_p be its connected components. For each i , select a vertex $v_i \in N(u)$ that is adjacent to a vertex in C_i , and apply the induction hypothesis on $G[C_i \cup \{v_i\}]$ with v_i . Either $\chi(G[C_i \cup \{v_i\}]) \leq f_{t-1}(\omega(G))$ or v_i is the endpoint of an induced path on $t - 1$ vertices in $G[C_i \cup \{v_i\}]$. In the latter case, by combining with u we obtain an

*I tried to add all the details you could possibly want, and more – please email if anything seems unclear.

[†]All mistakes strictly mine.

induced path on t vertices that has u as an endpoint, hence the conclusion. Therefore, for every i , we have $\chi(G[C_i \cup \{v_i\}]) \leq f_{t-1}(\omega(G))$. It follows that $\chi(G \setminus N[u]) \leq f_{t-1}(\omega(G))$. We observe now that $\omega(G[N(u)]) \leq \omega(G) - 1$. As a consequence, $\chi(G) \leq 1 + (f_t(\omega(G) - 1)) + (f_{t-1}(\omega(G)))$, hence the conclusion. \square

4 Triangle-free subgraph

Theorem 4 (Rödl '77). *For every k , there is p such that every graph G with $\chi(G) \geq p$ contains a triangle-free subgraph H with $\chi(H) \geq k$.*

Proof. We will in fact prove the following statement. For every k and q , there is $f(k, q)$ such that every graph G with $\chi(G) \geq f(k, q)$ either contains a triangle-free subgraph H with $\chi(H) \geq k$ or a clique on q vertices. This implies Theorem 4 since there is a triangle-free graph with chromatic number at least k , so it suffices to take q being at least its number of vertices.

We proceed by induction on q . The statement holds when $q = 1$, so we assume $q \geq 2$. Let G be a graph which contains no clique on q vertices and no triangle-free subgraph with chromatic number k . For every vertex $u \in V(G)$, we have by induction $\chi(G[N(u)]) \leq f(k, q - 1)$. For every u , let c_u be a proper colouring of $N(u)$ using at most $f(k, q - 1)$ colours. We order the vertices of G arbitrarily, and assign to every edge uv the colour of v in c_u if $v \prec u$ in the ordering. The resulting edge colouring may not be proper, however we note that every colour class induces a triangle-free subgraph of G , which is by assumption $(k - 1)$ -colourable. To conclude, we use the following easy lemma, applied iteratively to account for all $f(k, q - 1)$ colour classes.

Lemma 5. *The union G_3 of two graphs G_1 and G_2 satisfies $\chi(G_3) \leq \chi(G_1) \times \chi(G_2)$.*

As a consequence, we have $\chi(G) \leq (k - 1)^{f(k, q-1)}$, hence the existence of $f(k, q)$. \square