

CR16: Exam

January 28, 2021

To get full marks, answer three out of these five exercises¹. If you answer more, we will only take into account your three best ones. All lecture notes allowed. All theorems and lemmas discussed in the lectures can be used without a proof.

Exercise 1

Let $k \in \mathbb{N}$. Remember that P_3 is the path on three vertices.

1. Argue that there is some function f such that every graph G with no induced copy of P_3 satisfies $\chi(G) \leq f(\omega(G))$.
2. Can f be a polynomial function?
3. Argue that there is some function f_k such that if a graph G does not contain k pairwise non-adjacent induced copies of P_3 , then $\chi(G) \leq f_k(\omega(G))$.
4. Can f_k be a polynomial function?
5. Can f_k be the identity function for all k ?

Exercise 2

1. Recall Euler's formula for connected planar graphs².

¹Please send a scan in pdf format both to marthe.bonamy@u-bordeaux.fr and stephan.thomasse@ens-lyon.fr. Keep your original paper files (if any), in case we need to ask for a higher-quality scan.

²No proof needed

2. Let G be a connected planar graph with no cycle of length less than g . Using Euler's formula, provide an upper bound³ on $|E(G)|$ as a function of $|V(G)|$ and g .
3. Taking $g = 4$, derive from the bound obtained above that $K_{3,3}$ is not planar.
4. Using the bound for $g = 4$, argue that a planar triangle-free graph is 4-choosable⁴.

Exercise 3

1. Describe a bipartite graph that is not 10-choosable.
2. Using the discharging method, argue that every planar graph G with $\Delta(G) \geq 10000$ satisfies $\chi'_\ell(G) \leq \Delta(G)+1$. Hint for induction purposes: it suffices to argue that for every $k \geq 10000$, all planar graphs with $\Delta(G) \leq k$ satisfy $\chi'_\ell(G) \leq k + 1$.

Exercise 4

1. Argue that every tree admits an orientation of its edges such that for every vertex, $|d^+(u) - d^-(u)| \leq 1$.
2. Argue that every graph whose edges can be decomposed into a tree and a cycle admits an orientation of its edges such that for every vertex, $|d^+(u) - d^-(u)| \leq 1$.
3. Argue that every graph admits an orientation of its edges such that for every vertex, $|d^+(u) - d^-(u)| \leq 1$.
4. Derive that every bipartite graph G is $\frac{\Delta(G)+3}{2}$ -choosable.

Exercise 5

Let $t \in \mathbb{N}^*$. Argue that the class of graphs with no induced $K_{t,t}$ and no K_t has bounded VC-dimension.

³Hint: When g tends to infinity the number of edges should get closer to the number of edges in a tree.

⁴There is a short argument.