

# CR16: Homework 1

December 4, 2020

*The deadline is December 22nd<sup>1</sup>. Particular care should be given to the writing, as the grading will significantly depend on it.*

**Exercise 1** Let  $C_n$  be the cycle of length  $n$  and  $c$  be a proper 3-coloring of  $C$ . Show that the number of *fixing pairs* of  $c$ , i.e. pairs of vertices  $u, v$  at distance two in  $C_n$  with  $c(u) \neq c(v)$ , is equal to  $n$  modulo 2. Deduce that the generalized Mycielski construction starting with an odd cycle and adding  $k$  rounds of twins provide a 4-chromatic graph.

**Exercise 2** Let  $\varepsilon > 0$  be some fixed real. Show that given as input a graph  $G$  on  $n$  vertices with minimum degree at least  $\varepsilon n$ , one can decide in polynomial time if  $G$  is 3-colorable or not. Try to optimize the computational complexity of your algorithm.

**Exercise 3** Given as input a set  $\mathcal{D}$  of  $n$  unit disks in the plane, compute in polynomial time a maximum subset of  $\mathcal{D}$  consisting of pairwise intersecting disks. Provide a polytime approximation algorithm with some fixed ratio when  $\mathcal{D}$  consists of arbitrary disks in the plane. Any APX algo will suit, do not try to optimize the ratio, this is a long standing open problem!

**Exercise 4** Show that the sequence of Mycielski graphs starting with the edge  $M_2$  and letting  $M_{i+1} := M(M_i)$  is universal, i.e. every triangle-free graph is an induced subgraph of some  $M_i$ . Show also that every graph  $G$  satisfies that  $\chi(M(G)) = \chi(G) + 1$ . Is it true that if  $G$  is  $k$ -critical, then  $M(G)$  is  $k + 1$ -critical?

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<sup>1</sup>to be sent in pdf format both to `marthe.bonamy@u-bordeaux.fr` and `stephan.thomasse@ens-lyon.fr`

**Exercise 5** Show that the fractional chromatic number of a graph  $G$  is the maximum ratio  $|V(H)|/\alpha(H)$  for some blow-up of some subgraph  $H$  of  $G$ .

**Exercise 6** Given two graphs  $G$  and  $H$ , show that every graph  $Z$  such that  $Z \rightarrow G$  and  $Z \rightarrow H$  satisfies  $Z \rightarrow G \times H$ . Show that if  $G$  and  $H$  are both 3-chromatic, then  $G \times H$  is also 3-chromatic.

**Exercise 7** Show that every bipartite graph has a matching which covers all vertices with maximum degree. Deduce that line graphs of bipartite graphs are perfect.

**Exercise 8** Pick one of the following question for your homework (depending of your background/taste).

- (probabilistic method) Show the existence of graphs on  $n$  vertices with girth at least  $g$  and chromatic number  $k$ . For fixed  $g$  and  $k$  try to minimize the order of  $n$ .
- (basic topology) Show the equivalence between the four statements of Borsuk-Ulam given in the course.
- (infinite combinatorics) Using an ultraproduct, show that if all finite subgraphs of a graph  $G$  have chromatic number at most  $k$  then  $\chi(G) \leq k$ . Show that this is easily done when  $k = 2$  (with axiom of choice).
- (algorithms) Find a function  $f_3$  and a polynomial time algorithm which properly color any triangle-free graph  $G$  on  $n$  vertices with  $f_3(n)$  colors. Try to make  $f_3$  reasonably small. Find also  $f_4$ , when the input graph  $G$  has no 4-cycle (non necessarily induced). What about  $f_k$  in general?

**Exercise 9** Show without using Kempe chains that a minimum counter-example to the Five Colour Theorem does not contain a vertex of degree 5.

**Exercise 10** Show using only discharging<sup>2</sup> that all (non-empty) planar graphs contain a vertex of degree at most 4 or a vertex of degree 5 adjacent to a vertex of degree  $\leq 6$ .

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<sup>2</sup>without arguing that it suffices to handle the triangulated case.