

Adjacent vertex-distinguishing edge coloring of graphs*

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Abstract

An adjacent vertex-distinguishing edge coloring (AVD-coloring) of a graph is a proper edge coloring such that no two neighbors are adjacent to the same set of colors. Zhang *et al.* [17] conjectured that every connected graph on at least 6 vertices is AVD $(\Delta + 2)$ -colorable, where Δ is the maximum degree.

In this paper, we prove that $(\Delta + 1)$ colors are enough when Δ is sufficiently larger than the maximum average degree, denoted mad . We also provide more precise lower bounds for two graph classes: planar graphs, and graphs with $\text{mad} < 3$. In the first case, $\Delta \geq 12$ suffices, which generalizes the result of Edwards *et al.* [7] on planar bipartite graphs. No other results are known in the case of planar graphs. In the second case, $\Delta \geq 4$ is enough, which is optimal and completes the results of Wang and Wang [14] and of Hocquard and Montassier [9].

1 Introduction

In the following, a graph is a connected simple graph on at least three vertices. A (*proper*) *edge k -coloring* of a graph is a coloring of its edges using at most k colors, where any two incident edges receive distinct colors. The *chromatic index* of a graph G , denoted by $\chi'(G)$, is the smallest integer k such that G admits an edge k -coloring. Let $\Delta(G)$ be the maximum degree of G . Since incident edges receive distinct colors in an edge coloring, every graph G satisfies $\chi'(G) \geq \Delta(G)$. The Vizing's theorem ensures that the reverse inequality is nearly true, more precisely:

Theorem 1 [12] *Every graph G satisfies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

An *adjacent vertex-distinguishing edge k -coloring* (AVD k -coloring for short) is a proper edge k -coloring such that, for any two adjacent vertices u and v , the set of colors assigned to edges incident to u differs from the set of colors assigned to edges incident to v . The *AVD-chromatic index* of G , denoted by $\chi'_{avd}(G)$, is the smallest integer k such that G admits an AVD k -coloring. It should be noted that, while an isolated edge admits no AVD coloring, the AVD-chromatic index is finite for all connected graphs on at least three vertices. AVD colorings are also known as *adjacent strong edge colorings* [17] and *1-strong edge colorings* [1]. Note that AVD colorings are special cases of *vertex-distinguishing proper edge colorings*. Such colorings are proper edge colorings such that no two (not necessarily adjacent) vertices are adjacent to the same set of colors. The corresponding chromatic index is called the *observability* and was studied for different graph classes [3, 5, 6, 8].

Since an AVD coloring is a proper edge coloring, every graph G satisfies $\chi'_{avd}(G) \geq \Delta(G)$. In addition, every graph G with two adjacent vertices of degree $\Delta(G)$ satisfies $\chi'_{avd}(G) \geq \Delta(G) + 1$. Zhang *et al.* [17] completely determined the AVD-chromatic index of paths, cycles, trees, complete graphs, and complete bipartite graphs. They noted that a cycle of length five requires five colors, but conjectured that it is the only graph with such a gap between $\chi'_{avd}(G)$ and $\Delta(G)$.

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Conjecture 2 [17] *Every graph G on at least 6 vertices satisfies $\chi'_{avd}(G) \leq \Delta(G) + 2$.*

Balister *et al.* [2] proved Conjecture 2 for graphs with $\Delta(G) = 3$ and for bipartite graphs. For edge coloring, Theorem 1 ensures that the chromatic index of a graph can only have two values: $\Delta(G)$ or $\Delta(G) + 1$, and the classification of graphs depending on this received considerable interest (for instance [11]). For AVD coloring, if Conjecture 2 holds then the AVD chromatic index of a graph can only have three values: $\Delta(G)$, $\Delta(G) + 1$ or $\Delta(G) + 2$. When considering a given graph class that allows two vertices of maximum degree to be adjacent, there are only two possible upper bounds: $\Delta(G) + 1$ or $\Delta(G) + 2$. Similarly, the classification of graph classes depending on this received subsequent interest, for instance:

Theorem 3 [7] *Every (connected) bipartite planar graph G with $\Delta(G) \geq 12$ satisfies $\chi'_{avd}(G) \leq \Delta(G) + 1$.*

Let $\text{mad}(G) = \max \left\{ \frac{2|E(H)|}{|V(H)|}, H \subseteq G \right\}$ be the *maximum average degree* of the graph G , where $V(H)$ and $E(H)$ are the sets of vertices and edges of H , respectively. Wang and Wang [14] made the link between maximum average degree and AVD-chromatic index and proved Conjecture 2 for graphs with $\Delta(G) \geq 3$ and $\text{mad}(G) < 3$.

Theorem 4 [14] *Every (connected) graph G with $\Delta(G) \geq 3$ and $\text{mad}(G) < 3$ satisfies $\chi'_{avd}(G) \leq \Delta(G) + 2$.*

They also gave sufficient conditions for graphs of bounded maximum average degree to be AVD $(\Delta(G) + 1)$ -colorable. Combined with results of Hocquard and Montassier [9], we have:

Theorem 5 [9, 14] *Every (connected) graph G with $\Delta(G) \geq 3$ and $\text{mad}(G) < 3 - \frac{2}{\Delta(G)}$ satisfies $\chi'_{avd}(G) \leq \Delta(G) + 1$.*

Two main questions arise from these partial results: can this threshold of 3 as an upper-bound on $\text{mad}(G)$ be reached with a sufficiently large lower-bound on $\Delta(G)$ in the case of Theorem 5, and broken in the case of Theorem 4? We answer positively to these questions with Theorem 6: there is no threshold in the case of Theorem 5 (and thus in the case of Theorem 4).

Theorem 6 *Every graph G with $\Delta(G) > 3 \times (\text{mad}(G))^2$ satisfies $\chi'_{avd}(G) \leq \Delta(G) + 1$.*

In the case of edge coloring, the best lower bound is due to Woodall [15]: every graph G with $\Delta(G) > \frac{3 \times \text{mad}(G)}{2}$ satisfies $\chi'(G) = \Delta(G)$. There is a very large gap between this bound and its AVD counterpart, but this is essentially due to the fact that most methods on edge coloring are not transposable to AVD coloring. On the other hand, the gap between the bound for AVD coloring and its list edge counterpart is a mere constant factor [4] (note that list edge coloring is similarly conjectured to be always possible with $\Delta(G) + 1$ colors [13]).

By Theorem 6, planar graphs with sufficiently large maximum degree are AVD $(\Delta(G) + 1)$ -colorable. We provide a more refined lower-bound, and prove here that the bipartite hypothesis in Theorem 3 is unnecessary.

Theorem 7 *Every planar graph G with $\Delta(G) \geq 12$ satisfies $\chi'_{avd}(G) \leq \Delta(G) + 1$.*

In the case of graphs with maximum average degree at most 3, we improve Theorem 5 by showing that $\Delta(G) \geq 4$ is enough to reach the threshold of 3, as follows.

Theorem 8 *Every graph G with $\Delta(G) \geq 4$ and $\text{mad}(G) < 3$ satisfies $\chi'_{avd}(G) \leq \Delta(G) + 1$.*

Note that Theorem 8 is best possible since Figure 1 provides a subcubic graph with $\text{mad}(G) = \frac{11}{4} < 3$ that is not AVD $(\Delta + 1)$ -colorable.

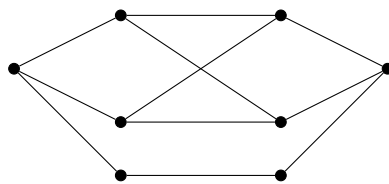


Figure 1: A graph G with $\Delta(G) = 3$ and $\text{mad}(G) = \frac{11}{4} < 3$ such that $\chi'_{avd}(G) = 5$.

2 Method

We prove Theorems 6, 7 and 8 using a discharging method. We first choose a partial order on graphs. It depends on the theorem to prove but is basically a customization of the lexicographic order on the number of vertices of each degree. In a first time, we consider by contradiction a “minimal” counter-example, and prove that it has some strong structural properties. We then prove that a graph with those structural properties cannot satisfy the assumptions of the theorem, which provides a contradiction. Due to the limited number of pages, the complete proofs of Theorems 6, 7 and 8 are omitted. We nevertheless sketch the proof of Theorem 8.

The four following lemmas provide the most relevant examples of structural properties that we proved on a minimal counter-example of any of Theorems 6, 7 and 8, where the number of colors is $k + 1$, for $k \geq 4$. In each case, we assume by contradiction that the graph contains such a configuration, we color by minimality a smaller graph and prove that the coloring can be extended to the whole graph.

Lemma 9 *No vertex v_2 is adjacent to two vertices v_1 and v_3 , with $d(v_1), d(v_2)$ and $d(v_3) \leq \frac{k}{2}$.*

Lemma 9 follows from a recoloring algorithm. The proof is quite involved, when a simple proof exists when $\frac{k}{2}$ is replaced by $\frac{k}{4}$, but this bound is decisive for Theorems 6 and 7.

Lemma 10 *No vertex has at least $\frac{k}{2}$ neighbors of degree 1.*

Lemma 11 *No vertex of degree 2 is adjacent to two vertices of degree at most $\frac{k}{2} + 1$.*

Lemmas 10 and 11 follow from a simple combinatorial argument, except in the case of a vertex of degree 2 adjacent to a vertex of degree exactly $\frac{k}{2} + 1$, where the result is derived from a recoloring argument and from the 2-connectivity of a minimal counter-example.

Lemma 12 *No vertex is adjacent to two vertices u, v with $d(u) = 2$ and $d(v) \leq 2$.*

Lemma 12 follows from a simple reduction to a smaller graph, where the choice of the partial order is decisive. Those four lemmas are not enough for Theorems 6 and 7. Lemma 9 is decisive for Theorems 6 and 7, and Lemmas 10 to 12 suffice for Theorem 8.

We consider a graph G with $\Delta(G) \leq k$ that satisfies Lemmas 10, 11 and 12, and assign to each vertex its degree as weight. We then design discharging rules to rearrange the weight along the graph so as to derive that $\text{mad}(G) \geq 3$. The following observation is instrumental in the proofs of Theorems 6 and 8.

Observation 1 *For any vertex partition (V_1, V_2) of a graph G , if every vertex v has an initial weight of $d(v)$, and the weight can be rearranged along the graph so that every vertex v_1 of V_1 has a weight of at least $2 \times d(v_1)$ and every vertex v_2 of V_2 has a weight of at least m , then $\text{mad}(G) \geq m$.*

Thus, to prove Theorem 8, we design a single discharging rule stating that a vertex u with $d(u) \geq 3$ that has a neighbor v of degree 1 or 2 gives a weight of 1 to v . We consider V_1 to be the set of vertices of degree 1, and V_2 the set of vertices of degree at least 2. By Lemma 10, the vertices incident to a vertex of degree 1 are of degree at least 4, and in particular V_1 is an independent set. Let u be a vertex of G . If $u \in V_1$, since V_1 is an independent set, the vertex u gives nothing and receives $d(u)$, so it has a final weight of $2 \times d(u)$. If $u \in V_2$ with $d(u) = 2$, by Lemma 11, the vertex u is adjacent to at least one vertex of degree at least 3, and receives 1 from it. Since it

gives nothing, it has a final weight of at least 3. If $u \in V_2$ with $d(u) \geq 3$ and u has a neighbor of degree 2, by Lemmas 11 and 12, the vertex u is adjacent to no other vertex of degree at most 2, and $d(u) \geq 4$, so u has a final weight of at least 3. If $u \in V_2$ and has no neighbor of degree 2 with $d(u) \geq 3$, by Lemma 10, the vertex u has at most $d(u) - 3$ neighbors of degree 1, so u has a final weight of at least 3. By Observation 1, $\text{mad}(G) \geq 3$.

For Theorem 7, we use a combinatorial argument to prove that a vertex cannot have too many small vertices, with an optimal bound (optimal for a combinatorial argument not involving any recoloring algorithm) depending on the respective degrees and on the number of colors. For Theorem 6, we use a method from a beautiful proof by Borodin, Kostochka and Woodall [4] (later simplified in [15]) of a similar result on list edge coloring.

3 Conclusion and perspectives

With Theorem 6, we made a significant step towards Conjecture 2, by proving that there are many graphs that need one less color. Our methods will however not be sufficient for the conjecture itself, as they require sparsity hypotheses. However, we could aim at proving that all planar graphs are AVD $(\Delta(G) + 2)$ -colorable, by further developing the proof of Theorem 7. We conclude with two conjectures.

Conjecture 13 *For any graph G , if the set of vertices of maximum degree in G forms an independent set, then $\chi'_{\text{avd}}(G) \leq \Delta(G) + 1$.*

Conjecture 14 *For any graph G , if G admits a subgraph H such that $\chi'_{\text{avd}}(H) > \chi'_{\text{avd}}(G)$, then either H is not connected or $\Delta(H) = 2$.*

References

- [1] S. Akbari, H. Bidkhori, and N. Nosrati, r -strong edge colorings of graphs, *Discrete Math.*, 306:3005-3010, 2006.
- [2] P. N. Balister, E. Győri, J. Lehel, and R. H. Schelp, Adjacent vertex distinguishing edge-colorings, *SIAM J. Discrete Math.*, 21:237-250, 2007.
- [3] P.N. Balister, O.M. Riordan, and R.H. Schelp, Vertex-distinguishing edge-colorings of graphs, *J. Graph Theory*, 42:95-109, 2003.
- [4] O. V. Borodin, A. V. Kostochka and D. R. Woodall, List Edge and List Total Colourings of Multigraphs, *J. Comb. Theory, Series B*, 71(2):184-204, 1997.
- [5] A.C. Burriss and R.H. Schelp, Vertex-distinguishing proper edge-colorings, *J. Graph Theory*, 26:73-83, 1997.
- [6] J. Cerný, M. Horňák, and R. Soták, Observability of a graph, *Mathematica Slovaca*, 46(1):21-31, 1996.
- [7] K. Edwards, M. Horňák, and M. Woźniak, On the Neighbour-Distinguishing Index of a Graph, *Graphs and Comb.*, 22(3):341-350, 2006.
- [8] O. Favaron, H. Li, and R.H. Schelp, Strong edge coloring of graphs. *Discrete Math.*, 159:103-109, 1996.
- [9] H. Hocquard and M. Montassier, Adjacent vertex-distinguishing edge coloring of graphs with maximum degree Δ , DOI: 10.1007/s10878-011-9444-9, 2012.
- [10] A. V. Kostochka and D. R. Woodall, Choosability conjectures and multicircuits, *Discrete Math.*, 240:123-143, 2001.

- [11] D. P. Sanders and Y. Zhao, Planar Graphs of Maximum Degree Seven are Class I, *J. Comb. Theory, Series B*, 83(2):201-212, 2001.
- [12] V.G. Vizing, On an estimate of the chromatic class of a p -graph, *Metody Diskret. Analiz.*, 3:23-30, 1964.
- [13] Vizing, V. G., Colouring the vertices of a graph with prescribed colours (in russian), *Diskret. Analiz.*,29:3-10, 1976.
- [14] W. Wang and Y. Wang, Adjacent vertex distinguishing edge-colorings of graphs with smaller maximum average degree, *J. Comb. Optim.*, 19:471-485, 2010.
- [15] D. R. Woodall, The average degree of an edge-chromatic critical graph II, *J. Graph Theory* 56(3):194-218, 2007.
- [16] D. R. Woodall, The average degree of a multigraph critical with respect to edge or total choosability, *Discrete Math.*, 310:1167-1171, 2010.
- [17] Z. Zhang, L. Liu, and J. Wang, Adjacent strong edge coloring of graphs, *Appl. Math. Lett.*, 15:623-626, 2002.