

# Graphs with $\text{mad} < 3$ and $\Delta \geq 17$ are list 2-distance $(\Delta + 2)$ -colorable

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EXTENDED ABSTRACT

All the graphs considered here are simple and finite. A *2-distance  $k$ -coloring* of a graph  $G$  is a coloring of the vertices of  $G$  with  $k$  colors such that two vertices that are adjacent or have a common neighbor receive distinct colors. We define  $\chi^2(G)$  as the smallest  $k$  such that  $G$  admits a 2-distance  $k$ -coloring. A generalization of the 2-distance  $k$ -coloring is the *list 2-distance  $k$ -coloring*, where instead of having the same list of  $k$  colors for the whole graph, every vertex is assigned some set of  $k$  colors and has to be colored from it. We define  $\chi_\ell^2(G)$  as the smallest  $k$  such that  $G$  admits a list 2-distance  $k$ -coloring of  $G$  for any list assignment. Obviously, 2-distance coloring is a sub-case of list 2-distance coloring (where the same color list is assigned to every vertex), so for any graph  $G$ ,  $\chi_\ell^2(G) \geq \chi^2(G)$ .

The study of  $\chi^2(G)$  on planar graphs was initiated by Wegner in 1977 [7], and has been actively studied because of his conjecture, stated below. The *maximum degree* of a graph  $G$  is denoted  $\Delta(G)$ .

**Conjecture 1 (Wegner [7])** *If  $G$  is a planar graph, then:*

- $\chi^2(G) \leq 7$  if  $\Delta(G) = 3$
- $\chi^2(G) \leq \Delta(G) + 5$  if  $4 \leq \Delta(G) \leq 7$
- $\chi^2(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$  if  $\Delta(G) \geq 8$

Note that any graph  $G$  satisfies  $\chi^2(G) \geq \Delta(G) + 1$ . Indeed, if we consider a vertex of maximal degree and its neighbors, they form a set of  $\Delta(G) + 1$  vertices, any two of which are adjacent or have a common neighbor. Hence at least  $\Delta(G) + 1$  colors are needed for a 2-distance coloring of  $G$ . It is therefore natural to ask when this lower bound is reached. For that purpose, we can study, as suggested by Wang and Lih [6], what conditions on the sparseness of the graph can be sufficient to ensure the equality holds. A first measure of the sparseness of a planar graph is its girth. The *girth* of a graph  $G$ , denoted  $g(G)$ , is the length of a shortest cycle.

**Conjecture 2 (Wang and Lih [6])** *For any integer  $k \geq 5$ , there exists an integer  $D(k)$  such that for every planar graph  $G$  verifying  $g(G) \geq k$  and  $\Delta(G) \geq D(k)$ ,  $\chi^2(G) = \Delta(G) + 1$ .*

Conjecture 2 was proved by Borodin, Ivanova and Noestroeveva [3, 4] to be true for  $k \geq 7$ , even in the case of list-coloring, and false for  $k \in \{5, 6\}$ .

Dvořák, Král, Nejedlý and Škrekovski [5] proved that it is off by just one for  $k = 6$ , i.e. for a planar graph  $G$  with girth 6 and sufficiently large  $\Delta(G)$ ,  $\chi^2(G) \leq \Delta(G) + 2$ . They also conjectured that the same holds for planar graphs with girth 5, but this remains open. Borodin and Ivanova [1, 2] improved the corresponding bound for graphs of girth 6, and extended it to list-coloring.

**Theorem 3 (Borodin and Ivanova [1])** *Every planar graph  $G$  with  $\Delta(G) \geq 18$  and  $g(G) \geq 6$  admits a 2-distance  $(\Delta(G) + 2)$ -coloring.*

**Theorem 4 (Borodin and Ivanova [2])** *Every planar graph  $G$  with  $\Delta(G) \geq 24$  and  $g(G) \geq 6$  admits a list 2-distance  $(\Delta(G) + 2)$ -coloring.*

We improve the previous two theorems as follows.

**Theorem 5** *Every planar graph  $G$  with  $\Delta(G) \geq 17$  and  $g(G) \geq 6$  admits a list 2-distance  $(\Delta(G) + 2)$ -coloring.*

Another way to measure the sparseness of a graph is through its maximum average degree as defined below. The *average degree* of a graph  $G$ , denoted  $\text{ad}(G)$ , is  $\frac{\sum_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|}$ . The *maximum average degree* of a graph  $G$ , denoted  $\text{mad}(G)$ , is the maximum of  $\text{ad}(H)$  over all subgraph  $H$  of  $G$ . Using this measure, we, in fact, prove a more general theorem than Theorem 5.

**Theorem 6** *Every graph  $G$  with  $\Delta(G) \geq 17$  and  $\text{mad}(G) < 3$  admits a list 2-distance  $(\Delta(G) + 2)$ -coloring.*

Euler's formula links girth and maximum average degree in the case of planar graphs, as it easy to check that for any planar graph  $G$ ,  $(\text{mad}(G) - 2)(g(G) - 2) < 4$ . Thus, planar graphs of girth at least 6 have a maximum average degree smaller than 3, and Theorem 5 is a corollary of Theorem 6.

To prove Theorem 6, we use a global discharging method, that is, a discharging method where some forbidden configurations have unbounded size and where the weight can travel arbitrarily far.

An *injective  $k$ -coloring* of  $G$  is a (not necessarily proper) coloring of the vertices of  $G$  with  $k$  colors such that no vertex has two neighbors with the same color, or, in other words, such that two vertices that have a common neighbor receive distinct colors. A 2-distance  $k$ -coloring is also an injective coloring, but the reverse is not true. The list version of this coloring is a *list injective  $k$ -coloring* of  $G$ .

Some results on 2-distance coloring have their counterpart on injective coloring with one less color, and it is the case of Theorems 3 and 4. It happens that the proof of Theorem 6 also works with close to no alteration for list injective coloring, thus yielding a proof that every graph  $G$  with  $\Delta(G) \geq 17$  and  $\text{mad}(G) < 3$  admits a list injective  $(\Delta(G) + 1)$ -coloring.

## References

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