Graphs with \( \text{mad} < 3 \) and \( \Delta \geq 17 \) are list 2-distance \((\Delta + 2)\)-colorable

Marthe Bonamy
Benjamin Lévêque
Alexandre Pinlou

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Vertex coloring

\[ \chi: \text{Minimum number of colors to ensure that } a \neq b. \]

\[ a \in L(x) \quad b \in L(y) \]

\[ \chi \ell: \text{Minimum size of every } L(v) \text{ such that } \Delta + 1 \leq \chi \leq \chi \ell. \]
\( \chi \): Minimum number of colors to ensure that

\[ a \neq b. \]
Vertex coloring

\[ \chi: \text{Minimum number of colors to ensure that} \]
\[ a \neq b. \]

\[ \chi_L: \text{Minimum size of every } L(v) \text{ such that} \]
\[ a \neq b \]
\[ a \in L(x) \]
\[ b \in L(y) \]
\( \chi^2: \) Minimum number of colors to ensure that

\[ a \sim b \text{ or } a \sim \circ \sim b \Rightarrow a \neq b. \]
2-distance coloring

\[ \chi^2: \text{Minimum number of colors to ensure that} \]
\[ a \neq b. \]

\[ \chi_\ell^2: \text{Minimum size of every } L(v) \text{ such that} \]
\[ \begin{align*}
(a \neq b) & \Rightarrow a \in L(x) \\
& \quad \text{and} \\
& \quad b \in L(y)
\end{align*} \]
\( \chi^2 \): Minimum number of colors to ensure that
\[
\begin{align*}
\fbox{a} & \quad \fbox{b} \quad \text{or} \quad \fbox{a} \quad \fbox{0} \quad \fbox{b} \quad \Rightarrow \quad a \neq b.
\end{align*}
\]

\( \chi^2_{\ell} \): Minimum size of every \( L(v) \) such that
\[
\begin{align*}
\fbox{a} & \quad \fbox{b} \quad \text{or} \quad \fbox{a} \quad \fbox{0} \quad \fbox{b} \quad \Rightarrow \quad \begin{cases} 
a \neq b \\
a \in L(x) \\
b \in L(y)
\end{cases}
\end{align*}
\]

\[ \Delta + 1 \leq \chi^2 \leq \chi^2_{\ell} \]
Two conjectures

Conjecture (Kostochka and Woodall 2001)

\[ \chi^2 = \chi^2_{\ell}. \]
Two conjectures

Conjecture (Kostochka and Woodall 2001)
\[ \chi^2 = \chi_{\ell}^2. \]

Conjecture (Wegner 1977)
For planar graphs, \( \chi^2 \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1 \) if \( \Delta \geq 8 \).
State of the art

\[ g : \text{girth} = \text{minimum length of a cycle.} \]
State of the art

\( g \) : girth = minimum length of a cycle.

**Question (Wang and Lih 2001)**

For planar graphs, \( g \geq k, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1 \).
State of the art

\( g \): girth = minimum length of a cycle.

**Question (Wang and Lih 2001)**

What is the minimal \( k \) such that for some \( d \),

For planar graphs, \( g \geq k, \Delta \geq d \) \( \Rightarrow \chi^2 = \Delta + 1 \).
State of the art

\( g \) : girth = minimum length of a cycle.

Question (Wang and Lih 2001)

What is the minimal \( k \mid \exists d \),
For planar graphs, \( g \geq k, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1 \).

Answer: \( k = 7 \). (Borodin, Ivanova and Noestroeva 2008)
State of the art

\( g \): girth = minimum length of a cycle.

**Question (Wang and Lih 2001)**

*What is the minimal \( k \) such that for planar graphs, \( g \geq k \), \( \Delta \geq d \) implies \( \chi^2 = \Delta + 1 \).*

**Answer:** \( k = 7 \). (Borodin, Ivanova and Noestroeva 2008)

- For planar graphs, \( \exists d, g \geq 7, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1 \).
- \( \exists (G_n)_{n \in \mathbb{N}}, \text{planar, } g = 6, \Delta \nearrow, \chi^2 > \Delta + 1 \).
**State of the art**

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**Theorem (Dvořák, Král, Nejedlý and Sřekovskii 2008)**

*For planar graphs, \( \exists d, g \geq 6, \Delta \geq d \Rightarrow \chi^2 \leq \Delta + 2. \)*
State of the art

\( g \) : girth = minimum length of a cycle.

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**Theorem (Borodin and Ivanova 2009)**

For planar graphs, \( g \geq 6, \Delta \geq 18 \Rightarrow \chi^2 \leq \Delta + 2 \).
State of the art

$g$ : girth = minimum length of a cycle.

**Question (Wang and Lih 2001)**

*What is the minimal $k$ such that $\exists d$, for planar graphs, $g \geq k$, $\Delta \geq d \Rightarrow \chi^2_{\ell} = \Delta + 1$.*

**Answer:** $k = 7$. (Borodin, Ivanova and Noestroeva 2008)

- For planar graphs, $\exists d$, $g \geq 7$, $\Delta \geq d \Rightarrow \chi^2_{\ell} = \Delta + 1$.
- $\exists (G_n)_{n \in \mathbb{N}}$, planar, $g = 6$, $\Delta \nearrow$, $\chi^2_{\ell} > \Delta + 1$.

**Theorem (Borodin and Ivanova 2009)**

*For planar graphs, $g \geq 6$, $\Delta \geq 1824 \Rightarrow \chi^2_{\ell} \leq \Delta + 2$.*
State of the art

g : girth = minimum length of a cycle.

Question (Wang and Lih 2001)

What is the minimal \( k \) such that \( \exists d \),
For planar graphs, \( g \geq k, \Delta \geq d \Rightarrow \chi_\ell^2 = \Delta + 1 \).

Answer: \( k = 7 \). (Borodin, Ivanova and Noestroeva 2008)

- For planar graphs, \( \exists d, g \geq 7, \Delta \geq d \Rightarrow \chi_\ell^2 = \Delta + 1 \).
- \( \exists (G_n)_{n \in \mathbb{N}}, \) planar, \( g = 6, \Delta \uparrow, \chi_\ell^2 > \Delta + 1 \).

Theorem (Borodin and Ivanova 2009)

For planar graphs, \( g \geq 6, \Delta \geq 18 \Rightarrow \chi_\ell^2 \leq \Delta + 2 \).

Theorem

For planar graphs, \( g \geq 6, \Delta \geq 17 \Rightarrow \chi_\ell^2 \leq \Delta + 2 \).
Another measure of sparsity

\[ \text{ad: average degree} = \frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|}. \]
Another measure of sparsity

**ad**: average degree = $\frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|}$.

**mad**: maximum average degree = $\max \frac{2|E(H)|}{|V(H)|}$.
Another measure of sparsity

ad: average degree = $\frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|}$. 

mad: maximum average degree = $\max \frac{2|E(H)|}{|V(H)|}$. 

Lemma (Derived from Euler’s Formula)

For any planar graph, $(\text{mad} - 2)(g - 2) < 4$
Another measure of sparsity

\[ \text{ad: average degree } = \frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|}. \]

\[ \text{mad: maximum average degree } = \max \frac{2|E(H)|}{|V(H)|}. \]

**Lemma (Derived from Euler’s Formula)**

*For any planar graph, \((\text{mad} - 2)(g - 2) < 4\)*

For planar graphs:

- \(g \geq 7 \Rightarrow \text{mad} < \frac{14}{5}\).
- \(g \geq 6 \Rightarrow \text{mad} < 3\).
Question

What is the \textbf{maximal} \( m \) \( \mid \exists d, \) 
For any graph, \( \text{mad} < m, \Delta \geq d \Rightarrow \chi^2_{\ell} = \Delta + 1. \)
Question

What is the maximal $m$ such that for any graph, $\text{mad} < m$, $\Delta \geq d \Rightarrow \chi_2^\ell = \Delta + 1$.

Answer: $m = \frac{14}{5}$?
Question

What is the maximal $m | \exists d$,

For any graph, $\text{mad} < m$, $\Delta \geq d \Rightarrow \chi_2^\ell = \Delta + 1$.

Answer: True for any $m < 3$.

- $\forall \epsilon > 0$, $\text{mad} < 3 - \epsilon$, $\Delta \geq O(\frac{1}{\epsilon^2}) \Rightarrow \chi_2^\ell = \Delta + 1$.

- $\exists (G_n)_{n \in \mathbb{N}}$, $\text{mad} < 3$, $\Delta \nearrow$, $\chi_2^\ell \geq \Delta + 1$. 
Question

What is the maximal $m$ | $\exists d$, For any graph, $\text{mad} < m$, $\Delta \geq d \Rightarrow \chi_2^\ell = \Delta + 1$.

Answer: True for any $m < 3$.

- $\forall \epsilon > 0$, $\text{mad} < 3 - \epsilon$, $\Delta \geq O\left(\frac{1}{\epsilon^2}\right) \Rightarrow \chi_2^\ell = \Delta + 1$.
- $\exists (G_n)_{n \in \mathbb{N}}$, $\text{mad} < 3$, $\Delta \nearrow$, $\chi_2^\ell > \Delta + 1$.

Theorem

For any graph, $\text{mad} < 3$, $\Delta \geq 17 \Rightarrow \chi_2^\ell \leq \Delta + 2$. 
Discharging methods

1. Every graph with $\text{mad} < 3$ contains some $C \in \{C_1, C_2, \ldots, C_p, \}$. 

Marthe Bonamy, Benjamin Lévêque, Alexandre Pinlou
List 2-distance $(\Delta + 2)$-coloring
1. Every graph with $\text{mad} < 3$ contains some $C \in \{C_1, C_2, \ldots, C_p\}$.

2. For the problem, a graph that contains a $C_i$ can be reduced to a smaller graph.
Discharging methods

1. Every graph with $\text{mad} < 3$ contains some $C \in \{C_1, C_2, \ldots, C_p\}$.

2. For the problem, a graph that contains a $C_i$ can be reduced to a smaller graph.

3. Thus every graph with $\text{mad} < 3$, $\Delta \geq 17$ is list 2-distance $(\Delta + 2)$-colorable.
Every graph with \( \text{mad} < 3 \) contains some \( C \) 
\[ \in \{ C_1, C_2, \ldots, C_p, \} \].
Mad formula: \( \text{mad}(G) < 3 \)

For the problem, a graph that contains a \( C_i \) can be reduced to a smaller graph.

Thus every graph with \( \text{mad} < 3, \Delta \geq 17 \) is list 2-distance \( (\Delta + 2) \)-colorable.
Discharging methods

1. Every graph with $\text{mad} < 3$ contains some $C \in \{C_1, C_2, \ldots, C_p\}$. 
   Mad formula: $\text{ad}(G) < 3$

2. For the problem, a graph that contains a $C_i$ can be reduced to a smaller graph.

3. Thus every graph with $\text{mad} < 3$, $\Delta \geq 17$ is list 2-distance $(\Delta + 2)$-colorable.
Discharging methods

1. Every graph with $\text{mad} < 3$ contains some $C \in \{C_1, C_2, \ldots, C_p\}$.
   Mad formula: $\sum_v (d(v) - 3) < 0$

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Discharging methods

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   ![](image)

   $\Rightarrow \forall v, w' \quad (v) \geq 0 \quad \Rightarrow \text{Contradiction!}$

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Discharging methods

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   Mad formula: $\sum_v (d(v) - 3) < 0$

   \[
   \begin{array}{c}
   \bullet \\
   (4,5) \\
   (4,5) \\
   \end{array}
   \]

   $\Rightarrow \forall v, w'(v) \geq 0 \Rightarrow \text{Contradiction!}$

2. For the problem, a graph that contains a $C_i$ can be reduced to a smaller graph.

3. Thus every graph with $\text{mad} < 3, \Delta \geq 17$ is list 2-distance $(\Delta + 2)$-colorable.
Global discharging method

No induced cycle with even number of edges in the graph:
- No intersecting cycles (Cactus)
- Multiplicity at most 2
- On every cycle, at least two edges of multiplicity 1

ad < 4 \mid \{|\{\}\} \leq 2 \times \{|\{\}\|
Global discharging method

Structure induced by $\bigcirc \longrightarrow \bigcirc$ in the graph:
Global discharging method

Structure induced by \( \circ \rightarrow \circ \) in the graph:
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Global discharging method

Structure induced by \( \circ \longrightarrow \circ \) in the graph:

- No induced cycle with even number of \( \bullet \)
- \( \Rightarrow \) multigraph on \( \circ \):
Global discharging method

Structure induced by \(\circ - \circ\) in the graph:
- No induced cycle with even number of \(\bullet\)
- \(\Rightarrow\) multigraph on \(\circ\):
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Global discharging method

Structure induced by $\bigcirc - \bigcirc$ in the graph:

- No induced cycle with even number of $\bullet$
- $\Rightarrow$ multigraph on $\bigcirc$:
  - No intersecting cycles (Cactus)
  - Multiplicity at most 2
  - On every cycle, at least two edges of multiplicity 1
  - $ad < 4$
  - $|\{\bullet\}| \leq 2 \times |\{\bigcirc\}|$
Thanks for your attention.

Any questions?
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