

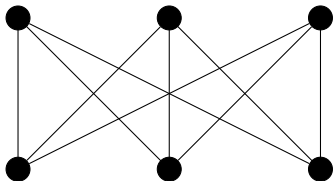
Graphs with $\text{mad} < 3$ and $\Delta \geq 17$
are list 2-distance $(\Delta + 2)$ -colorable

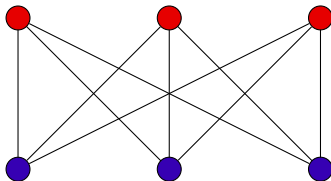
Marthe Bonamy
Benjamin Lévêque
Alexandre Pinlou

November 21, 2012



Vertex coloring

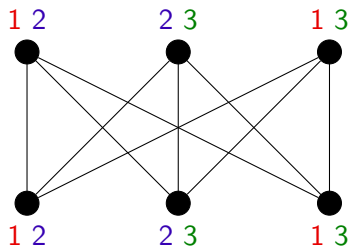




χ : Minimum number of colors to ensure that

$$\textcircled{a} \text{---} \textcircled{b} \Rightarrow a \neq b.$$

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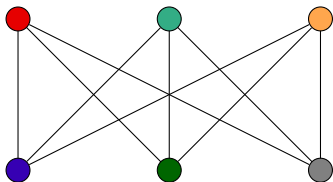


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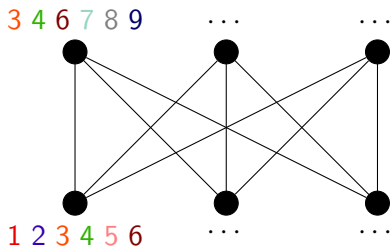
$$\textcircled{a} \text{---} \textcircled{b} \Rightarrow \begin{cases} a \neq b \\ a \in L(x) \\ b \in L(y) \end{cases}$$



χ^2 : Minimum number of colors to ensure that

$$\textcircled{a} \text{---} \textcircled{b} \text{ or } \textcircled{a} \text{---} \textcirc{} \text{---} \textcircled{b} \Rightarrow a \neq b.$$

2-distance coloring



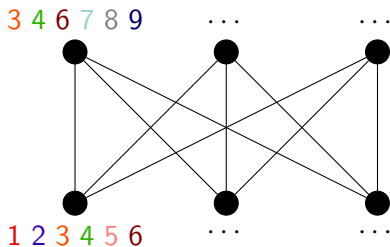
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$$\Delta + 1 \leq \chi^2 \leq \chi_\ell^2$$

Two conjectures

Conjecture (Kostochka and Woodall 2001)

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Conjecture (Wegner 1977)

For *planar* graphs, $\chi^2 \leq \lfloor \frac{3\Delta}{2} \rfloor + 1$ if $\Delta \geq 8$.

State of the art

g : girth = minimum length of a cycle.

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Question (Wang and Lih 2001)

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- For *planar* graphs, $\exists d, g \geq 7, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1$.
- $\exists (G_n)_{n \in \mathbb{N}}$, *planar*, $g = 6, \Delta \nearrow, \chi^2 > \Delta + 1$.

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Theorem (Dvořák, Král, Nejedlý and Škrekovski 2008)

For *planar* graphs, $\exists d, g \geq 6, \Delta \geq d \Rightarrow \chi^2 \leq \Delta + 2$.

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Theorem (Borodin and Ivanova 2009)

For *planar* graphs, $g \geq 6, \Delta \geq 18 \Rightarrow \chi^2 \leq \Delta + 2$.

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Theorem

For *planar* graphs, $g \geq 6$, $\Delta \geq 17 \Rightarrow \chi_\ell^2 \leq \Delta + 2$.

Another measure of sparsity

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For any *planar* graph, $(\text{mad} - 2)(g - 2) < 4$

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For *planar* graphs:

- $g \geq 7 \Rightarrow \text{mad} < \frac{14}{5}$.
- $g \geq 6 \Rightarrow \text{mad} < 3$.

Question

What is the *maximal* $m \mid \exists d$,
For *any* graph, $\text{mad} < m$, $\Delta \geq d \Rightarrow \chi_{\ell}^2 = \Delta + 1$.

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Answer: True for any $m < 3$.

- $\forall \epsilon > 0$, $\text{mad} < 3 - \epsilon$, $\Delta \geq \mathcal{O}(\frac{1}{\epsilon^2}) \Rightarrow \chi_\ell^2 = \Delta + 1$.
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Mad and list 2-distance coloring

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Theorem

For any graph, $\text{mad} < 3$, $\Delta \geq 17 \Rightarrow \chi_\ell^2 \leq \Delta + 2$.

- 1 Every graph with $\text{mad} < 3$ contains some $\mathcal{C} \in \{C_1, C_2, \dots, C_p, \}$.

Discharging methods

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- 2 For the problem, a graph that contains a C_i can be reduced to a smaller graph.
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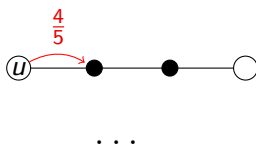
Discharging methods

- 1 Every graph with $\text{mad} < 3$ contains some \mathcal{C}
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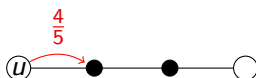


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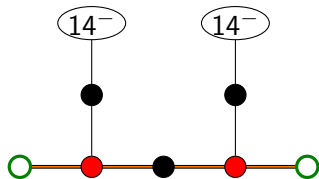
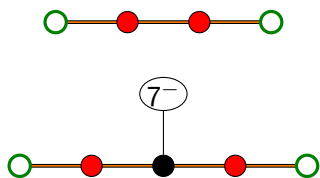
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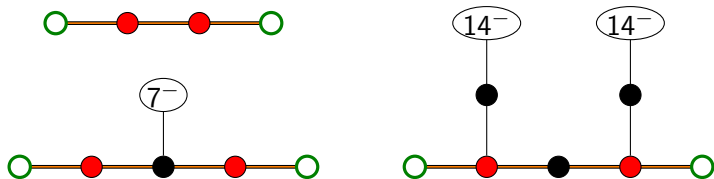
$\Rightarrow \forall v, w'(v) \geq 0 \Rightarrow$ **Contradiction!**

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Global discharging method

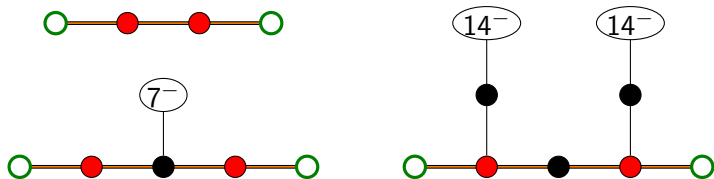


Global discharging method



Structure induced by $\circ - \circ$ in the graph:

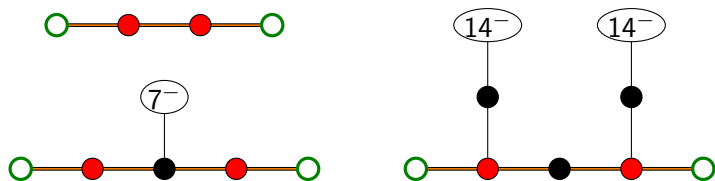
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- No induced cycle with even number of \bullet

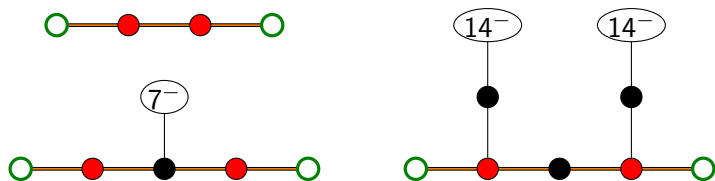
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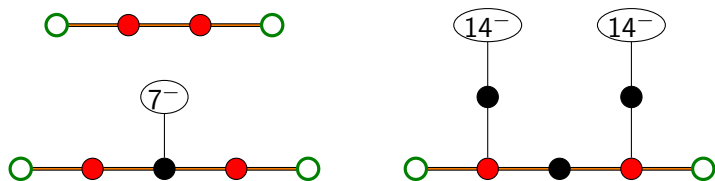
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 - On every cycle, at least two edges of multiplicity 1

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- No induced cycle with even number of \bullet
- \Rightarrow multigraph on O :
 - No intersecting cycles (Cactus)
 - Multiplicity at most 2
 - On every cycle, at least two edges of multiplicity 1
 - $\text{ad} < 4$
- $|\{\bullet\}| \leq 2 \times |\{\text{O}\}|$

Conclusion

Thanks for your attention.
Any questions?