2-distance coloring of sparse graphs

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August 31, 2011
2-distance coloring

χ: Minimum number of colors to ensure that $a \neq b$.

χ₂: Minimum number of colors to ensure that $a \neq b$ or $a \neq b$.

\[ \chi_2 \geq \Delta + 1. \]
$\chi$: Minimum number of colors to ensure that

\[ a \neq b. \]
2-distance coloring

\[ \chi: \text{Minimum number of colors to ensure that} \]

\[ a \neq b. \]

\[ \chi^2: \text{Minimum number of colors to ensure that} \]

\[ a \neq b. \]
2-distance coloring

\( \chi \): Minimum number of colors to ensure that

\( a \rightarrow b \Rightarrow a \neq b \).

\( \chi^2 \): Minimum number of colors to ensure that

\( a \rightarrow \overline{b} \text{ or } a \rightarrow b \Rightarrow a \neq b \).

\( \Delta \): Maximum degree of the graph.

\( \chi^2 \geq \Delta + 1 \).
A graph is **planar** if it can be drawn in a plane without crossing edges.

### Conjecture (Wegner 1977)

If $G$ is a planar graph, then:

- $\chi_2^2 \leq 7$ if $\Delta = 3$
- $\chi_2^2 \leq \Delta + 5$ if $4 \leq \Delta \leq 7$
- $\chi_2^2 \leq \left\lfloor \frac{3\Delta}{2} \right\rfloor + 1$ if $\Delta \geq 8$
2-distance coloring

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✓ C. Thomassen
Sparse graphs

$$\text{ad: average degree} = \frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|}.$$
Sparse graphs

\[
\text{ad: average degree} = \frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|}.
\]

\[
\text{mad: maximum average degree} = \max \frac{2|E(H)|}{|V(H)|}.
\]
Sparse graphs

ad: average degree = $\frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|}$.

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g : girth = minimum length of a cycle.
Sparse graphs

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\[ g : \text{ girth } = \text{ minimum length of a cycle.} \]

**Lemma (Derived from Euler's Formula)**

*For any planar graph, \((\text{mad} - 2)(g - 2) < 4\)*
Sparse graphs

ad: average degree $= \frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|}$.

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Lemma (Derived from Euler’s Formula)
For any planar graph, $(\text{mad} - 2)(g - 2) < 4$

$g \geq k, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1.$
Sparse graphs

ad: average degree = \( \frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|} \).

mad: maximum average degree = \( \max \frac{2|E(H)|}{|V(H)|} \).

g: girth = minimum length of a cycle.

Lemma (Derived from Euler’s Formula)

For any planar graph, \((\text{mad} - 2)(g - 2) < 4\)

In the case of planar graphs,

\[ \exists k_0, \forall k \geq k_0, \exists d, \]

\[ g \geq k, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1. \]
Theorem (Borodin, Ivanova, Neustroeva ’08 & Ivanova ’11)

For any planar graph, \( \chi^2 = \Delta + 1 \) in each of the following cases:

- \( g \geq 24 \) and \( \Delta \geq 3 \)
- \( g \geq 15 \) and \( \Delta \geq 4 \)
- \( g \geq 12 \) and \( \Delta \geq 5 \)

Theorem (Dolama, Sopena ’05)

For any graph, if \( \Delta \geq 4 \) and \( mad < 16 \), then \( \chi^2 = \Delta + 1 \).
State of the Art

Theorem (Borodin, Ivanova, Neustroeva ’08 & Ivanova ’11)

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Theorem (Dolama, Sopena ’05)

For any graph, if \( \Delta \geq 4 \) and \( \text{mad} < \frac{16}{7} \), then \( \chi^2 = \Delta + 1 \).
### Theorem (Borodin, Ivanova, Neustroeva ’08 & Ivanova ’11)

For any planar graph, $\chi^2 = \Delta + 1$ in each of the following cases:

- $g \geq 24$ and $\Delta \geq 3$
- $g \geq 15$ and $\Delta \geq 4$
- $g \geq 12$ and $\Delta \geq 5$
- $g \geq 10$ and $\Delta \geq 6$
- $g \geq 8$ and $\Delta \geq 10$
- $g \geq 7$ and $\Delta \geq 16$

### Theorem (Dolama, Sopena ’05)

For any graph, if $\Delta \geq 4$ and $\text{mad} < \frac{16}{7}$, then $\chi^2 = \Delta + 1$.

$\Rightarrow g \geq 16$
Our results

Theorem

For any graph, if $\Delta \geq 4$ and mad $< \frac{7}{3}$, then $\chi^2 = \Delta + 1$. 

The upper-bound on mad is optimal. $\Delta = 4$. mad $= \frac{7}{3}$. $\chi^2 = 6$. 

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Our results

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\[ \Delta = 4. \]
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For any graph, if $\Delta \geq 4$ and $\text{mad} < \frac{7}{3}$, then $\chi^2 = \Delta + 1$.

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\[ \Delta = 4. \]
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Our results

**Theorem**

*For any graph, if $\Delta \geq 4$ and $\text{mad} < \frac{7}{3}$, then $\chi^2 = \Delta + 1$.*

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$\Delta = 4.$

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Theorem

For any graph, if $\Delta \geq 4$ and $\text{mad} < \frac{7}{3}$, then $\chi^2 = \Delta + 1$.

The upper-bound on mad is optimal.

$\Delta = 4$.  
$\text{mad} = \frac{7}{3}$.  
$\chi^2 = 6$. 

Diagram: 

- Nodes colored in different colors to illustrate the coloring. 
- The graph shows connections between nodes to represent the graph structure.
Our results

**Theorem**

*For any graph, if $\Delta \geq 4$ and $\text{mad} < \frac{7}{3}$, then $\chi^2 = \Delta + 1$.  

$\Rightarrow g \geq 14$*

The upper-bound on mad is optimal.

$\Delta = 4.$  
$\text{mad} = \frac{7}{3}.$  
$\chi^2 = 6.$
Our results

\[ mad < m, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1. \]
Our results

\[ M \text{ maximal} \mid \forall m < M, \exists d, \]
\[ mad < m, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1. \]
Our results

Let $M$ be maximal such that $\forall m < M,$ there exists $d$ such that $mad < m,$ $\Delta \geq d \Rightarrow \chi^2 = \Delta + 1.$

**Theorem**

For a small enough $\epsilon,$ $mad < \frac{14}{5} - \epsilon$ and $\Delta \geq O\left(\frac{1}{\epsilon}\right) \Rightarrow \chi^2 = \Delta + 1.$
Our results

\[ M \text{ maximal } | \forall m < M, \exists d, \]
\[ mad < m, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1. \]

**Theorem**

*For a small enough \( \epsilon \), \( mad < \frac{14}{5} - \epsilon \) and \( \Delta \geq O\left(\frac{1}{\epsilon}\right) \Rightarrow \chi^2 = \Delta + 1. \)*

\[ \Delta = p. \]
\[ mad = 3 - \frac{5}{2p+1}. \]
\[ \chi^2 = \Delta + 2. \]
Our results

\[ M \text{ maximal } | \forall m < M, \exists d, \]
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Theorem

For a small enough \( \epsilon \), \( mad < \frac{14}{5} - \epsilon \) and \( \Delta \geq O\left(\frac{1}{\epsilon}\right) \Rightarrow \chi^2 = \Delta + 1. \)

\[ \Delta = p. \]
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\[ \chi^2 = \Delta + 2. \]

What happens between \( \frac{14}{5} \) and 3?
Our results

Corollary

For any planar graph, $\chi^2 = \Delta + 1$ in each of the following cases:

- $g(G) \geq 12$ and $\Delta(G) \geq 5$
- $g(G) \geq 10$ and $\Delta(G) \geq 6$
- $g(G) \geq 9$ and $\Delta(G) \geq 10$
Our results

\[ \text{mad} < m \Rightarrow \chi^2 \leq \Delta + C. \]
Our results

\[ M \text{ maximal } | \forall m < M, \exists C, \]
\[ mad < m \Rightarrow \chi^2 \leq \Delta + C. \]
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\[ M \text{ maximal } \mid \forall m < M, \exists C, \]
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**Theorem**

*For any* \( \epsilon > 0 \), \( \text{mad} < 4 - \epsilon \Rightarrow \chi^2 \leq \Delta + O(\frac{1}{\epsilon}) \).
Our results

\[ M \text{ maximal } \mid \forall \ m < M, \ \exists \ C, \]
\[ \text{mad} < m \Rightarrow \chi^2 \leq \Delta + C. \]

**Theorem**

*For any \( \epsilon > 0, \) \( \text{mad} < 4 - \epsilon \Rightarrow \chi^2 \leq \Delta + O\left(\frac{1}{\epsilon}\right). \)

\[ \Delta = p. \]
\[ \text{mad} = 4 - \frac{4}{p}. \]
\[ \chi^2 = \frac{3\Delta}{2}. \]
Our results

\[ M \text{ maximal } | \ \forall m < M, \ \exists C, \]
\[ \text{mad} < m \Rightarrow \chi^2 \leq \Delta + C. \]

**Theorem**

For any \( \epsilon > 0 \), \( \text{mad} < 4 - \epsilon \) \( \Rightarrow \chi^2 \leq \Delta + \mathcal{O}(\frac{1}{\epsilon}). \)

\[ \Delta = p. \]
\[ \text{mad} = 4 - \frac{4}{p}. \]
\[ \chi^2 = \frac{3\Delta}{2}. \]

Optimal theorem

\( M = 4 \)
Using a discharging method

A “semi-global” discharging method.
Using a discharging method

A “semi-global” discharging method.

- Assume there is a minimal counter-example
Using a discharging method

A “semi-global” discharging method.

- Assume there is a minimal counter-example
- Initial weight of a vertex: its degree
A “semi-global” discharging method.

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- Initial weight of a vertex: its degree
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- Discharging rules and Forbidden configurations
Using a discharging method

A “semi-global” discharging method.

- Assume there is a minimal counter-example
- Initial weight of a vertex: its degree
- Explicit feeding areas ($\approx$ trees)
- Discharging rules and Forbidden configurations
- Final weight of a vertex: $\geq \frac{7}{3}$
Using a discharging method

A “semi-global” discharging method.

- Assume there is a minimal counter-example
- Initial weight of a vertex: its degree
- Explicit feeding areas ($\simeq$ trees)
- Discharging rules and Forbidden configurations
- Final weight of a vertex: $\geq \frac{7}{3}$
- Contradiction
Using a discharging method

Forbidden Configurations

\[
\begin{align*}
1^- &\quad (k-1)^- \\
3^- &\quad 3^- \\
T &\quad T
\end{align*}
\]
Using a discharging method

Discharging Rules

\[ T \rightarrow \frac{1}{3} \]
\[ \neg T \rightarrow \frac{1}{6} \]

\[ 3 \rightarrow \frac{1}{6} \]
\[ 4 \rightarrow \frac{1}{3} \]

\[ 4^+ \rightarrow \frac{1}{6} \]
\[ 4^+ \rightarrow \frac{1}{6} \]
Conjecture (Kostochka, Woodall '01)

For any graph, $\chi^2 = \chi^2_\ell$. 
Conclusion

Conjecture (Kostochka, Woodall '01)

For any graph, $\chi^2 = \chi^2_\ell$.

Thanks for your attention.
Any questions?