

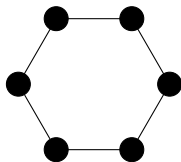
2-distance coloring of sparse graphs

Marthe Bonamy
Benjamin Lévêque
Alexandre Pinlou

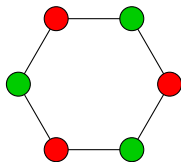
August 31, 2011



2-distance coloring

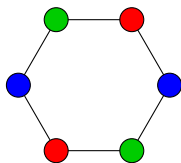


2-distance coloring



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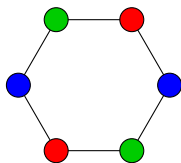


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Δ : Maximum degree of the graph.

$$\chi^2 \geq \Delta + 1.$$

A graph is **planar** if it can be drawn in a plane without crossing edges.

Conjecture (Wegner 1977)

If G is a *planar* graph, then:

- $\chi^2 \leq 7$ if $\Delta = 3$
- $\chi^2 \leq \Delta + 5$ if $4 \leq \Delta \leq 7$
- $\chi^2 \leq \lfloor \frac{3\Delta}{2} \rfloor + 1$ if $\Delta \geq 8$

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✓ C. Thomassen

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In the case of *planar* graphs,

$\exists k_0, \forall k \geq k_0, \exists d,$

$g \geq k, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1.$

Theorem (Borodin, Ivanova, Neustroeva '08 & Ivanova '11)

For any *planar* graph, $\chi^2 = \Delta + 1$ in each of the following cases :

- $g \geq 24$ and $\Delta \geq 3$
- $g \geq 15$ and $\Delta \geq 4$
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For any graph, if $\Delta \geq 4$ and $\text{mad} < \frac{16}{7}$, then $\chi^2 = \Delta + 1$.

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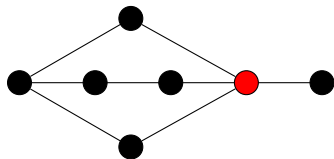
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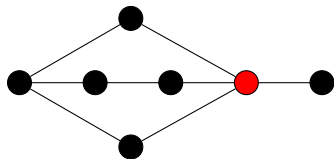


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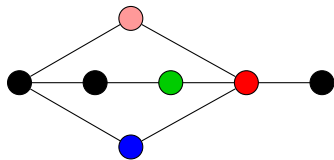


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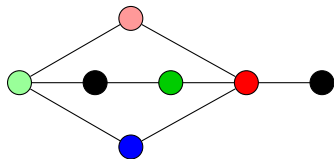


$$\begin{aligned}\Delta &= 4. \\ \text{mad} &= \frac{7}{3}. \\ \chi^2 &= 6.\end{aligned}$$

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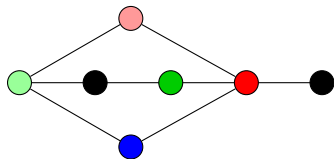
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$\rightsquigarrow g \geq 14$

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$$mad < m, \Delta \geq d \Rightarrow \chi^2 = \Delta + 1.$$

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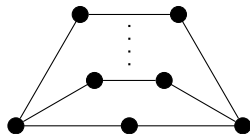
For a small enough ϵ , $mad < \frac{14}{5} - \epsilon$ and $\Delta \geq \mathcal{O}(\frac{1}{\epsilon}) \Rightarrow \chi^2 = \Delta + 1$.

Our results

M maximal $\mid \forall m < M, \exists d,$
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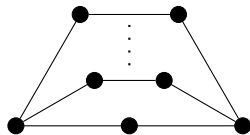
$$\begin{aligned}\Delta &= p. \\ \text{mad} &= 3 - \frac{5}{2p+1}. \\ \chi^2 &= \Delta + 2.\end{aligned}$$

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What happens between $\frac{14}{5}$ and 3?

$$\begin{aligned}\Delta &= p. \\ mad &= 3 - \frac{5}{2p+1}. \\ \chi^2 &= \Delta + 2.\end{aligned}$$

Corollary

For any *planar* graph, $\chi^2 = \Delta + 1$ in each of the following cases:

- $g(G) \geq 12$ and $\Delta(G) \geq 5$
- $g(G) \geq 10$ and $\Delta(G) \geq 6$
- $g(G) \geq 9$ and $\Delta(G) \geq 9$

$$\text{mad} < m \Rightarrow \chi^2 \leq \Delta + C.$$

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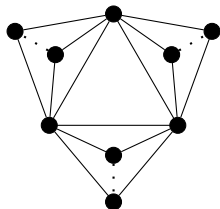
Theorem

For any $\epsilon > 0$, $mad < 4 - \epsilon \Rightarrow \chi^2 \leq \Delta + \mathcal{O}(\frac{1}{\epsilon}).$

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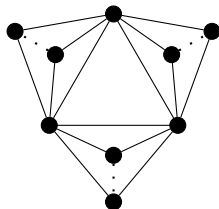


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Optimal theorem
 $M = 4$

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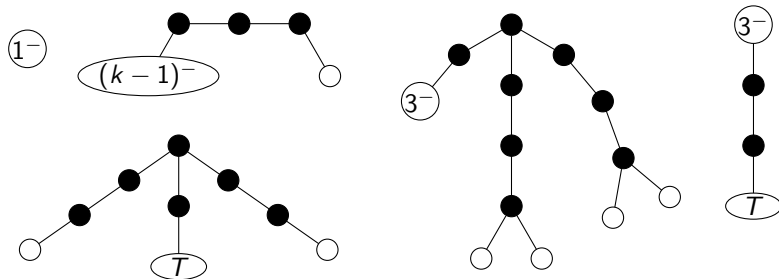
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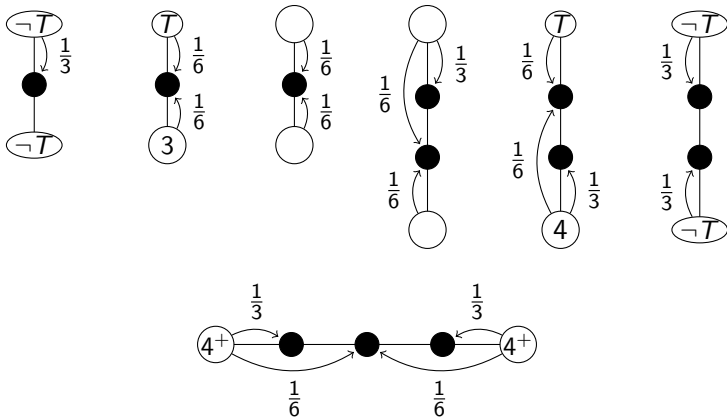
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- Contradiction

Forbidden Configurations



Discharging Rules



Conjecture (Kostochka, Woodall '01)

For any graph, $\chi^2 = \chi_\ell^2$.

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Thanks for your attention.
Any questions?