FPT meets discharging

Marthe Bonamy

July 10th, 2014

Based on joint work with
Łukasz Kowalik (University of Warsaw)
NP-hard graph problem
Fixed Parameter Tractability

NP-hard graph problem

FPT w.r.t. a parameter $k$ if:

- $(G, k)$ with $|V(G')| \leq f(k)$
- $|V(G')| \leq f(k)$
**Fixed Parameter Tractability**

**NP-hard** graph problem

**FPT w.r.t. a parameter** \( k \) if:

\[(G, k) \]

\[\xrightarrow{\text{Poly}(n)}\]

\[(G', k') \text{ with } |V(G')| \leq f(k)\]
Fixed Parameter Tractability

**NP-hard** graph problem

FPT w.r.t. a parameter $k$ if:

$(G, k)$

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(kernel)
Fixed Parameter Tractability

NP-hard graph problem
FPT w.r.t. a parameter $k$ if:

$$(G, k)$$

with

$$|V(G')| \leq f(k)$$

(kernel)

$$(G', k')$$

Bounded $k \Rightarrow \smiley$$
Fixed Parameter Tractability

NP-hard graph problem
FPT w.r.t. a parameter $k$ if:

$$(G, k)$$

$$(G', k')$$ with $|V(G')| \leq f(k)$
(kernel)

Bounded $k \Rightarrow 😊$

Best $f$?
Discharging methods

Just a counting argument.
Discharging methods

Just a counting argument.

$$1 + 2 + 3 + 4 + 5 + 6$$
Discharging methods

Just a counting argument.

$1+2+3+4+5+6$
Discharging methods

Just a counting argument.

$0 + 0 + 0 + 7 + 7 + 7$
Discharging methods

Just a counting argument.

\[0 + 0 + 0 + 7 + 7 + 7\]

**Euler’s formula for planar graphs**

\[|E| - |V| - |F| < 0\]
Discharging methods

Just a counting argument.

\[0 + 0 + 0 + 7 + 7 + 7\]

Euler's formula for planar graphs

\[|E| - |V| - |F| < 0\]

\[\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) < 0\]
<table>
<thead>
<tr>
<th><strong>Kernelization</strong></th>
<th><strong>Discharging in coloring</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph decision Problem $L$</td>
<td>Graph property $P$</td>
</tr>
</tbody>
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Find $G' \preceq G$, $k' \leq k$, $L(G', k') \iff L(G, k)$

Prove reduced graph has at most $f(k)$ vertices

Worked for planar non-blocker (Kowalik '12).
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**Prove reduced graph has at most $f(k)$ vertices**

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**FPT meets discharging**
Kernelization vs Discharging

**Kernelization**
Graph decision Problem $L$

1. Find $G' \prec G$, $k' \leq k$, $L(G', k') \Leftrightarrow L(G, k)$
2. Prove reduced graph has at most $f(k)$ vertices

**Discharging in coloring**
Graph property $P$

1. Find $G' \prec G$, $P(G') \Rightarrow P(G)$
2. Prove reduced graph has no vertex
## Kernelization vs Discharging

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1. Find $G' \prec G$, $k' \leq k$, $L(G', k') \iff L(G, k)$
2. Prove reduced graph has at most $f(k)$ vertices
   - Sunflowers

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Kernelization vs Discharging

**Kernelization**
Graph decision Problem \( L \)

1. Find \( G' \prec G, k' \leq k, L(G', k') \iff L(G, k) \)
2. Prove reduced graph has at most \( f(k) \) vertices
   - Sunflowers
   - Crown decomposition
   - Protusions

**Discharging in coloring**
Graph property \( P \)

1. Find \( G' \prec G, P(G') \Rightarrow P(G) \)
2. Prove reduced graph has no vertex
   - Discharging
   - Discharging
   - ...More discharging
Kernelization vs Discharging

**Kernelization**
Graph decision Problem $L$

1. Find $G' \prec G$, $k' \leq k$, $L(G', k') \iff L(G, k)$
2. Prove reduced graph has at most $f(k)$ vertices
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**Discharging in coloring**
Graph property $P$

1. Find $G' \prec G$, $P(G') \implies P(G)$
2. Prove reduced graph has no vertex
   - Discharging
   - Discharging
   - ...More discharging

Worked for planar non-blocker (Kowalik '12).
Feedback Vertex Set

Smallest $|S|$ $|G[V \setminus S]|$: forest?
Feedback Vertex Set

Smallest $|S|$, $|G[V \setminus S]|$: forest?

$\exists S, |S| \leq k$, $G[V \setminus S]$: forest?
Feedback Vertex Set

Smallest $|S|$ \ $|G[V \setminus S]|$: forest?

$\exists S, |S| \leq k, G[V \setminus S]$: forest?

One of Karp’s 21 NP-hard problems.
Feedback Vertex Set

Smallest $|S|$ $|G[V \setminus S]|$: forest?

$\exists S, |S| \leq k, G[V \setminus S]$: forest?

One of Karp’s 21 NP-hard problems.

**Theorem (Thomassé ’10)**

*FVS parameterized with the size of the solution: $\exists 4k^2$ kernel.*
Question

What about planar FVS?
State of the Art

Question

What about planar FVS?

- \( \exists c \), admits a \( c \cdot k \) kernel (Fomin, Lokshtanov, Saurabh, Thilikos '10)
State of the Art

Question

What about planar FVS?

- $\exists c$, admits a $c \cdot k$ kernel (Fomin, Lokshtanov, Saurabh, Thilikos '10)
- $c \leq 112$ (Bodlaender, Penninx '08)
State of the Art

Question

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- \( c \leq 97 \) (Abu-Khzam, Khuzam '12)
State of the Art

Question

What about planar FVS?

- $\exists c$, admits a $c \cdot k$ kernel (Fomin, Lokshtanov, Saurabh, Thilikos '10)
- $c \leq 112$ (Bodlaender, Penninkx ’08)
- $c \leq 97$ (Abu-Khzam, Khuzam ’12)
- $c \leq 13$ (B., Kowalik ’14+)

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Theorem (B., Kowalik’ 14+)  
*Planar FVS admits a 13k kernel.*

Proof sketch:
Theorem (B., Kowalik’ 14+)

Planar FVS admits a $16k$ kernel.

Proof sketch:
Sketch of the proof

Theorem (B., Kowalik’ 14+)

Planar FVS admits a 16k kernel.

Proof sketch:

- Easy reductions.
Theorem (B., Kowalik’ 14+)

**Planar FVS** admits a \(16k\) kernel.

Proof sketch:

- Easy reductions.

\[ \begin{array}{c}
  \text{Reduced graph?} \\
  \text{Return NO if more than 16k vertices.}
\end{array} \]
Theorem (B., Kowalik’ 14+)

Planar FVS admits a 16k kernel.

Proof sketch:

- Easy reductions.

```
    k   ▷ k
    ▷   ▷
```

Reduced graph?

Return NO if more than 16k vertices.
Theorem (B., Kowalik’ 14+)

**Planar FVS admits a 16k kernel.**

Proof sketch:

- Easy reductions.

\[ k \xrightarrow{} k \xrightarrow{} k \xrightarrow{} k \xrightarrow{} k \]

\[ k \xrightarrow{} k \xrightarrow{} k \xrightarrow{} k - 2 \]
Theorem (B., Kowalik’ 14+)

Planar FVS admits a 16k kernel.

Proof sketch:

- Easy reductions.

\[ k \rightarrow k \]

\[ k \rightarrow \]

\[ k \rightarrow k \]

- Reduced graph?

\[ \sim \sim \]
Theorem (B., Kowalik’ 14+)  

Planar FVS admits a 16k kernel.

Proof sketch:

- Easy reductions.

\[ \begin{align*}
  k \rightarrow & \quad k \\
  k \rightarrow & \quad k \\
  k \rightarrow & \quad k \\
  k \rightarrow & \quad k - 2 \\
  \end{align*} \]

- Reduced graph?
  Return NO if more than 16k vertices.
Contribution

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FPT meets discharging

8/10
Tight example
Conclusion

- Actual discharging?
- Other problems: Dominating Set...
Conclusion

- Actual discharging?
- Other problems: Dominating Set...

Thanks!