

# Reconfiguring Graph Homomorphisms and Colourings

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Joint work with  
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SIAM DM16

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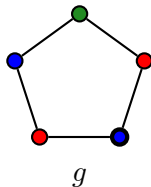
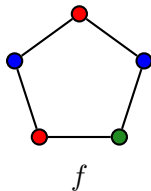
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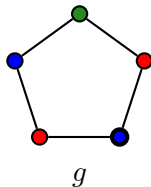
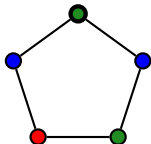
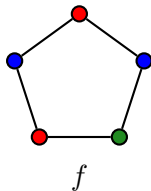
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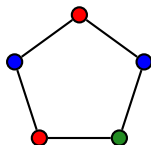
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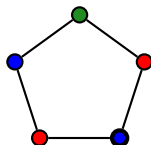
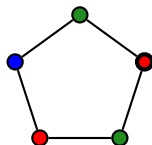
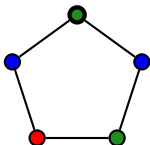
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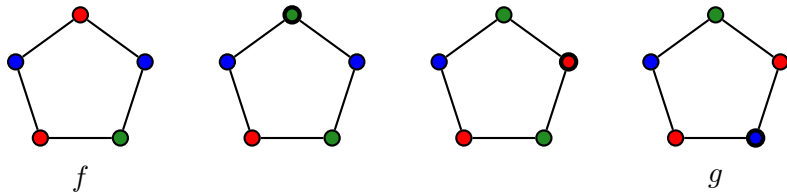
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**Our Focus:** In this talk, we are interested in the **complexity** of problems of this type.

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It is known that  $NP \subseteq PSPACE$  and *believed* that the inclusion is strict.

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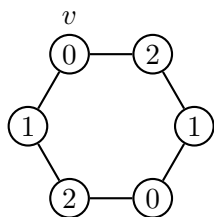
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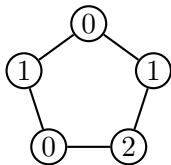
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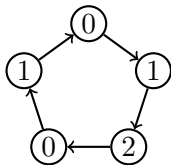
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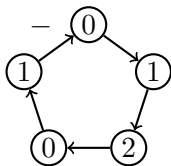
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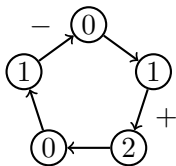
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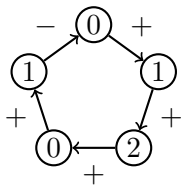
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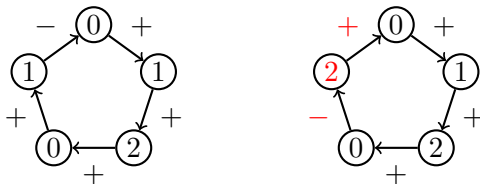
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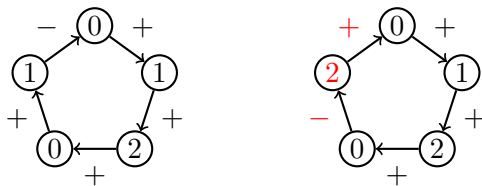




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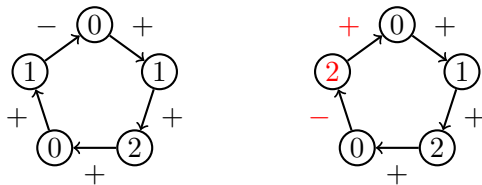


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**Theorem** (CvdHJ (2011)). These two conditions are also sufficient and can be checked in polynomial time.

# The Bigger Picture: Graph Homomorphisms

**Definition:** A **homomorphism** from a graph  $G$  to a graph  $H$  is a mapping  $f : V(G) \rightarrow V(H)$  such that  $f(u)f(v) \in E(H)$  for every  $uv \in E(G)$ .

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# Circular Colourings

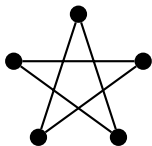
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The graph  $G_{p,q}$  is referred to as a **circular clique**.

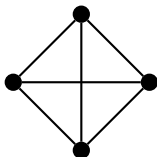
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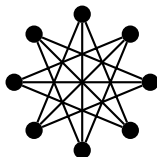
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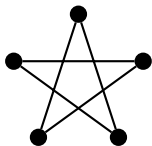


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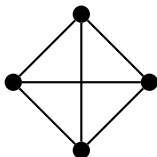
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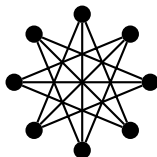
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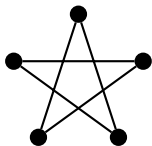
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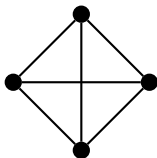
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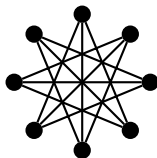
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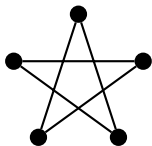
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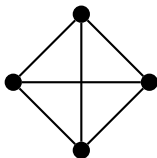
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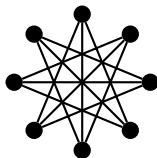
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Circular colourings were introduced by Vince (1988).

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For  $p/q \geq 4$ , we reduce the reconfiguration problem for  $\lfloor p/q \rfloor$ -colourings to the reconfiguration problem for  $(p, q)$ -colourings and apply the result of BC (2009).

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# Complexity of $H$ -RECOLOURING

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$K_2$	P	Trivial Exercise
$K_3$	P	CvdHJ (2011)
$C_4$ -free $H$	P	Wrochna (2015)
$G_{p,q}$ , $2 \leq p/q < 4$	P	BMMN (2016+)
$K_k$ , $k \geq 4$	PSPACE-complete	BC (2009)
$G_{p,q}$ , $p/q \geq 4$	PSPACE-complete	BMMN (2016+)
odd wheels, etc	PSPACE-complete	BMMN (2016+++)

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**Question:** Is the  $H$ -RECOLOURING problem PSPACE-complete for every graph  $H$  containing a  $K_4$ ?

**Question:** Is it true that the  $H$ -RECOLOURING problem is always either in P or PSPACE-complete? (That is, is there a dichotomy?)