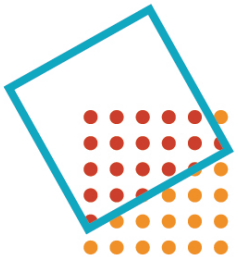
An overview of discharging methods

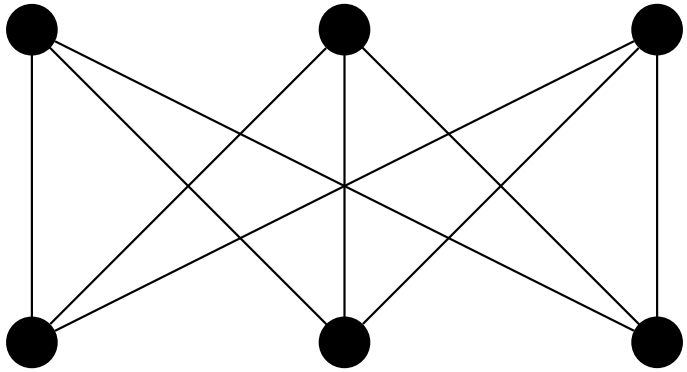
Marthe Bonamy

May 6th, 2014



Laboratoire
Informatique
Robotique
Microélectronique
Montpellier

Vertex coloring



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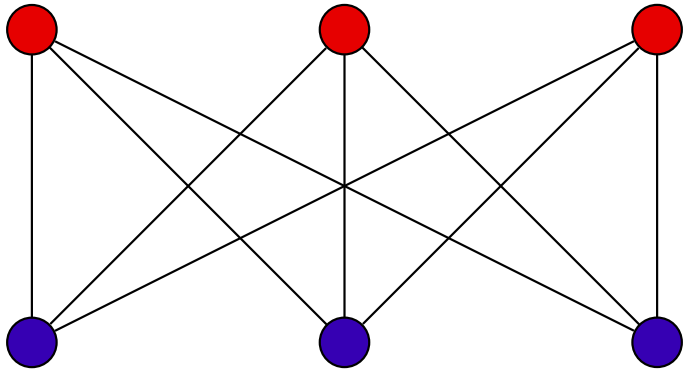

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$$\text{a} \text{---} \text{b} \Rightarrow a \neq b.$$

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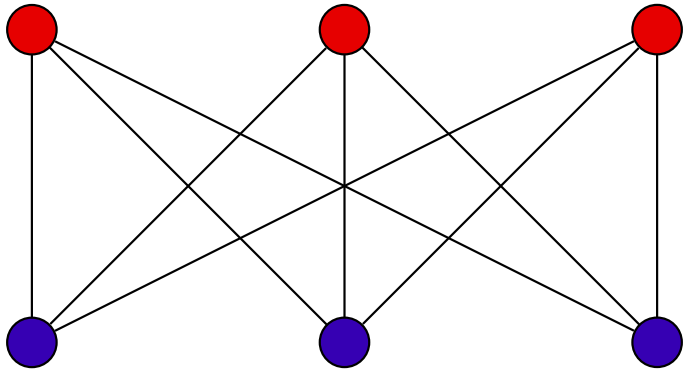

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A graph is planar if it can be drawn in a plane without crossing edges.

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Conjecture (Guthrie 1852)

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   Euler's formula: $|E| - |V| - |F| = -2$ (Proof)

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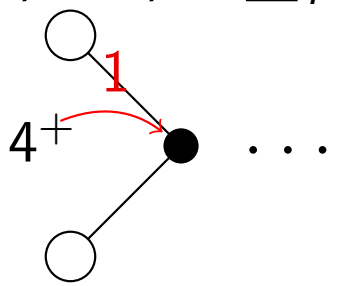

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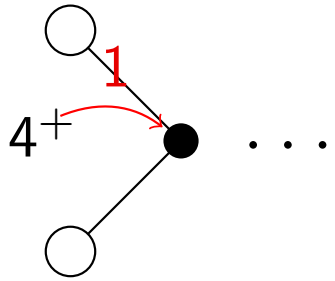

   $\Rightarrow \forall v, f, \ w'(v) \geq 0, w'(f) \geq 0$ Contradiction!

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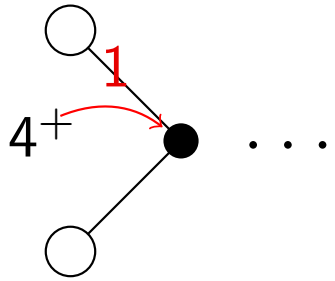

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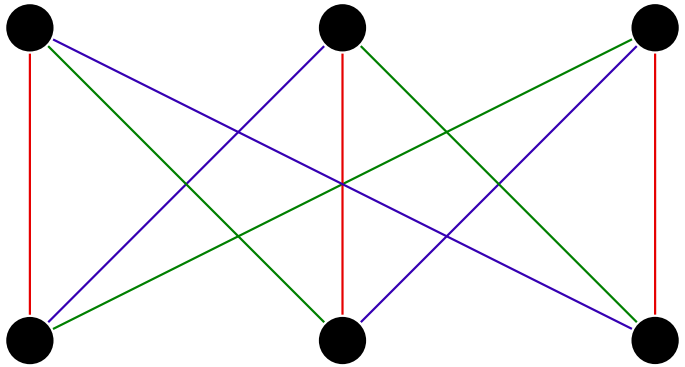
(Proof)

Edge coloring



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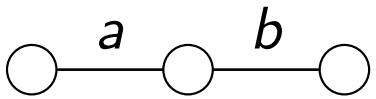

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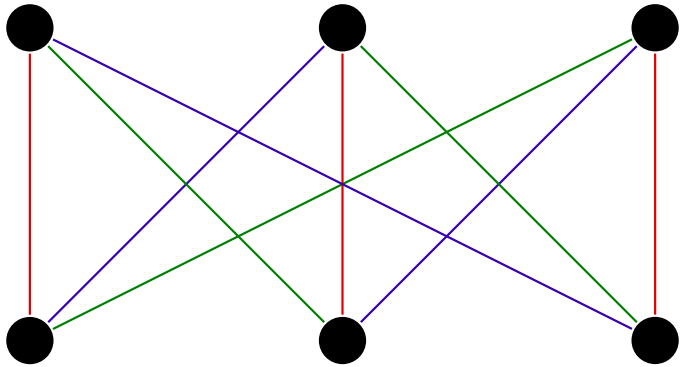

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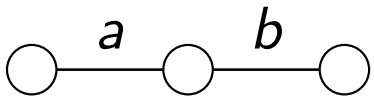

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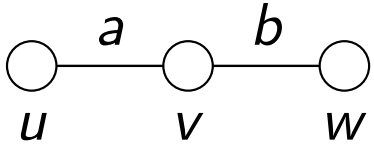
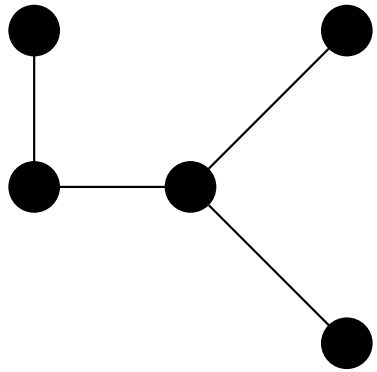
χ'_ℓ: Minimum size of every $L(e)$ such that



$$\Rightarrow \begin{cases} a \neq b \\ a \in L(u,v) \\ b \in L(v,w) \end{cases}$$

Edge Coloring (2)

Vertex coloring of Line graphs.



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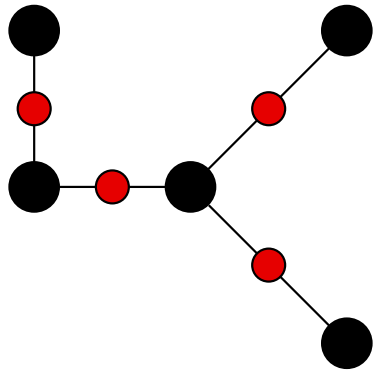

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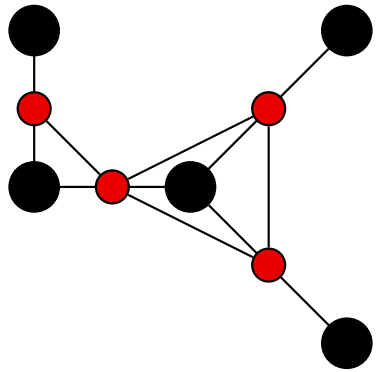

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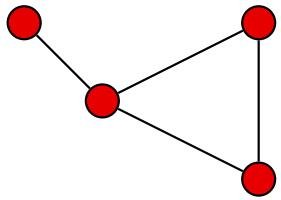

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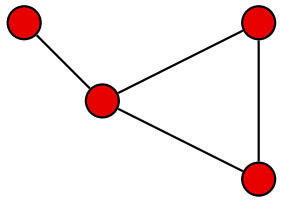

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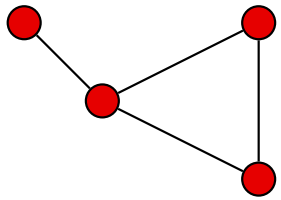

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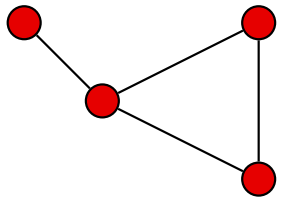

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Theorem (Vizing '64)

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On planar graphs:

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(Should be true for every single graph)

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Key lemma to the previous proof (and many other):

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Theorem (Borodin, Kostochka, Woodall '97)

For any graph, $\Delta > mad^2 \Rightarrow \chi'_\ell = \Delta$.

Can also be used in many similar settings.

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- But we should keep in mind it's just a counting argument (the world is not divided into discharging and non-discharging proofs).

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Thanks!