Bounded spectrum list coloring

Marthe Bonamy
19/09/2013

Joint work with Ross Kang
**Vertex coloring**

A graph is $k$-colorable if:

$$a \neq b, \quad a \in L(x), \quad b \in L(y)$$

A graph is $k$-choosable if:

For every $L$ with $|L(v)| \geq k$,
Graph $k$-colorable if:

$$a \not= b \quad a, b \in [1, \ldots, k]$$
Vertex coloring

Graph $k$-colorable if:

\[
(a) \not\equiv (b) \Rightarrow \begin{cases} 
    a \neq b \\
    a, b \in [1, \ldots, k]
\end{cases}
\]

Graph $k$-choosable if:

For every $L$ with $|L(v)| \geq k$,

\[
(a) \not\equiv (b) \Rightarrow \begin{cases} 
    a \neq b \\
    a \in L(x) \\
    b \in L(y)
\end{cases}
\]
Graph \((k, p)\)-choosable if:

For every \(L\) with \(|L(v)| \geq k\) and \(|\bigcup L(v)| \leq p\),

\[
\begin{align*}
\begin{array}{c}
 a \\
 x
\end{array} \quad \begin{array}{c}
 b \\
 y
\end{array} & \Rightarrow \\
\quad & \begin{cases}
 a \neq b \\
 a \in L(x) \\
 b \in L(y)
\end{cases}
\end{align*}
\]
Graph \((k, p)\)-choosable if:

For every \(L\) with \(|L(v)| \geq k\) and \(|\bigcup L(v)| \leq p\),

\[
\begin{align*}
\begin{cases}
  a \neq b \\
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\end{align*}
\]
Bounded spectrum list coloring

Graph \((k, p)\)-choosable if:

For every \(L\) with \(|L(v)| \geq k\) and \(|\bigcup L(v)| \leq p\),

\begin{align*}
\exists \begin{cases}
a \neq b \\
a \in L(x) \\
b \in L(y)
\end{cases}
\end{align*}

\((k, k)\)-choosable \iff k-colorable.
\((k, p + 1)\)-choosable \Rightarrow (k, p)\)-choosable.
\((k, \infty)\)-choosable \iff k-choosable.
Not (2, 4)-choosable, but (2, 3)-choosable.
Bipartite graphs

Not (2, 4)-choosable, but (2, 3)-choosable.

Not (2, 3)-choosable.
Bipartite graphs (2)

Lemma (Easy)

Any bipartite graph is \((k + 1, 2k)\)-choosable.
Bipartite graphs (2)

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Some bipartite graphs are not \((k + 1, 2k + 1)\)-choosable.
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Some bipartite graphs are not \((k + 1, 2k + 1)\)-choosable.

Lemma (Easy)

\((2, 3)\)-choosable \(\Rightarrow\) 2-choosable.
\((2, 4)\)-choosable \(\Rightarrow\) 2-choosable.
Lemma (Easy)

Any $\chi$-colorable graph is $(k(\chi - 1) + 1, k\chi)$-choosable.

Lemma (Easy)

Some bipartite graphs are not $(k + 1, 2k + 1)$-choosable.

Lemma (Easy)

$(2, 3)$-choosable $\not\Rightarrow$ 2-choosable.
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Lemma (Easy)

\((2, 3)\)-choosable \(\not\Rightarrow\) 2-choosable.
\((2, 4)\)-choosable \(\Rightarrow\) 2-choosable.
Questions

**Question**

\[ \exists \, C_k, (k, C_k)\text{-choosable} \Rightarrow k\text{-choosable}. \]

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Theorem (Král' Sgall 2005)

- For \( k \geq 3 \) and any \( p \geq k \), \((k, p)\)-choosable \( \Rightarrow k\)-choosable.

- And not \((k, p+1)\)-choosable.

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Marthe Bonamy

Bounded spectrum list coloring

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Questions

Question

$\exists \ C_k, (k, C_k)$-choosable $\Rightarrow$ $k$-choosable.

$k = 1 \checkmark$

$k = 2 \checkmark$
## Questions

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<th>$\exists \ C_k, (k, C_k)$-choosable $\implies k$-choosable.</th>
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**Theorem (Král’ Sgall 2005)**

For $k \geq 3$ and any $p \geq k$, $(k, p)$-choosable $\Rightarrow k$-choosable.
Questions

Question

∃? C_k, (k, C_k)-choosable ⇒ k-choosable.

k = 1 ✓

k = 2 ✓

Theorem (Král’ Sgall 2005)

For k ≥ 3 and any p ≥ k, there is a graph (k, p)-choosable and not (k, p + 1)-choosable.
Questions

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∃? \( C_k, (k, C_k)\)-choosable \( \Rightarrow \) \( k\)-choosable.

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Question (Král’ Sgall 2005)

∃? \( D_k, (k, D_k)\)-choosable \( \Rightarrow \) \( (k + 1)\)-choosable.

True for \( k = 2 \) (then \( D_2 = 3 \)).
Questions (2)

Question

∃? \( C_{k,p}, (k, p) \)-choosable \( \Rightarrow \) \( C_{k,p} \)-choosable.
∃? $C_{k,p}$, $(k, p)$-choosable $\Rightarrow C_{k,p}$-choosable.

$p \leq 2k - 2 : \text{NO}$
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For any $p \geq 2k - 1$, $C_{k,p} > \exp\left(\frac{(k-1)^2}{p}\right)$.

Only interesting if $p = o(k^2)$. 
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Definition (Berstein 1908)

A family of sets \( \mathcal{F} \subseteq \binom{[p]}{k} \) has property B if there is an intersecting set \( S \) of \( \mathcal{F} \) with \( F \not\subseteq S \) for all \( F \in \mathcal{F} \).
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(2-colorable $k$-uniform $p$-hypergraphs).
Properties B and K

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Hypergraphs?
**Property B**

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Property B

Definition (Berstein 1908)
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$M_{k,2k-1} = \binom{2k-1}{k}$
**Property B**

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$$M_{k,2k-1} = \binom{2k-1}{k}$$

**Theorem (Král’ Sgall 2005)**

$$\delta \geq 110 \cdot M(k, p)^2 \cdot \ln(M(k, p)) \Rightarrow \text{not } (k, p)\text{-choosable.}$$

Kang 2012: generalization to improper choosability.
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### Property K

**Definition (B. Kang 2013)**

A family of sets $\mathcal{F} \subseteq \left(\binom{[p]}{k}\right)$ has property K if there is a set $S \in \binom{[p]}{p-k+1}$ with $F \not\subseteq S \ \forall F \in \mathcal{F}$.

**Definition**

$R_{k,p} = \text{size of the smallest } \mathcal{F} \subseteq \left(\binom{[p]}{k}\right) \text{ that does not have property K (i.e. every set } S \in \binom{[p]}{p-k+1} \text{ contains an element of } \mathcal{F})$. 

---

**Theorem (B. Kang 2013)**

$R_{k,p} \geq \exp\left(\left(\frac{k-1}{2p}\right)^2\right)$. 

**Sketch of the proof (blackboard).**

$R_{k,p} \geq k^2 = k$. 

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**Definition (B. Kang 2013)**

A family of sets $\mathcal{F} \subseteq \binom{[p]}{k}$ has **property K** if there is a set $S \in \binom{[p]}{p-k+1}$ with $F \nsubseteq S \ \forall F \in \mathcal{F}$.

$$R_{k,2k-1} = \binom{2k-1}{k}.$$
Property K

**Definition (B. Kang 2013)**

A family of sets $\mathcal{F} \subseteq \binom{[p]}{k}$ has property K if there is a set $S \in \binom{[p]}{p-k+1}$ with $F \not\subseteq S \ \forall F \in \mathcal{F}$.

$R_{k,2k-1} = \binom{2k-1}{k}$.

**Lemma**

Any bipartite graph $(A, B)$ is $(k, p)$-choosable if $|A| < R(k, p)$.

**Theorem (B. Kang 2013)**

$$R_{k,p} \geq \exp\left(\frac{(k-1)^2}{p}\right).$$

Sketch of the proof (blackboard).

$R_{k,\geq k^2} = k$. 
Conclusion

Question (Král’ Sgall 2005)

∃? \( D_k, (k, D_k) \)-choosable \( \Rightarrow (k + 1)\)-choosable.

\[ D_k \overset{?}{=} k^2. \]
Conclusion

Question (Král’ Sgall 2005)

∃? \( D_k, (k, D_k) \)-choosable \( \Rightarrow (k + 1) \)-choosable.

- \( D_k \overset{?}{=} k^2 \).
- \( \delta \geq f(R_{k,p}) \overset{?}{\Rightarrow} \text{not } (k, p) \)-choosable.
Conclusion

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- \( D_k \stackrel{?}{=} k^2 \).
- \( \delta \geq f(R_k, p) \Rightarrow \) not \( (k, p) \)-choosable.
- \( (k, p) \)-choosability of complete bipartite graphs?
Question (Král’ Sgall 2005)

\[ \exists? D_k, (k, D_k)\text{-choosable} \Rightarrow (k + 1)\text{-choosable}. \]

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- \( \delta \geq f(R_k, p) \stackrel{?}{\Rightarrow} \) not \((k, p)\text{-choosable}\).
- \((k, p)\text{-choosability of complete bipartite graphs}\).