

Using discharging methods For the coloring of sparse graphs

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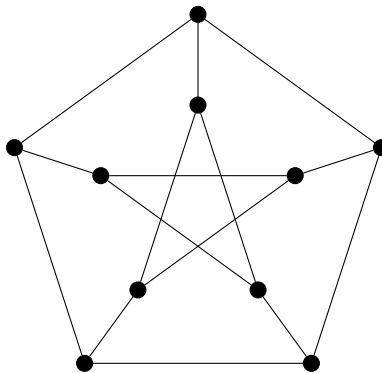
*M2 internship
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Abstract

The aim of this internship was to try a variation of the discharging method on some coloring problems and see whether it could yield more powerful results. We concentrated mainly on the 2-distance coloring of sparse graphs, and were able to improve known results by Dolama and Sopena, and Borodin et al.



Contents

1	Introduction	3
2	2-distance coloring	3
2.1	State of the Art	3
2.2	Our results	5
3	Discharging methods	6
3.1	A brief introduction	6
3.2	Using local discharging	8
3.3	Using semi-local discharging	10
4	Future work	15
	Addendum	15
	References	16
A	Proof of Theorem 4	17
A.1	Proof of Lemma 4	17
A.2	Proof of Lemma 5	20
A.3	Proof of Lemma 6	20
B	Proof of Theorem 5	21
B.1	Proof of Lemma 7	21
B.2	proof of Lemma 8	23
B.3	Proof of Lemma 9	23

1 Introduction

There are two main aspects to this internship: the problem we study, and the methods we use to solve it. In Section 2, we shall introduce the 2-distance coloring, state our main results and put them in context. In Section 3, we shall present the discharging method, give some simple examples, then sketch our proofs.

All the graphs we consider here are simple, finite and undirected. Let $G = (V, E)$ be a graph. For any subgraph H of G , we denote $V(H)$ and $E(H)$ the vertices and edges of H . For any vertex $v \in V$, the *degree* of v in G , denoted $d(v)$, is the number of neighbors of v in G . The *maximum degree* of G , denoted $\Delta(G)$, is $\max_{v \in V} d(v)$.

2 2-distance coloring

Definition 1. A 2-distance coloring of a graph G is a coloring of the vertices of G such that two vertices that are adjacent or have a common neighbor receive distinct colors. We define $\chi^2(G)$ as the smallest k such that G admits a 2-distance k -coloring.

This is equivalent to a proper vertex-coloring of the square of G , which is defined as a graph with the same set of vertices as G , where two vertices are adjacent if and only if they are adjacent or have a common neighbor in G . For example, a cycle of length 5 cannot be 2-distance colored with less than 5 colors as any two vertices are either adjacent or have a common neighbor.

2.1 State of the Art

The study of $\chi^2(G)$ on planar graphs was initiated by Wegner in 1977 [12], and has been actively studied because of his conjecture.

Conjecture 1 (Wegner [12]). *If G is a planar graph, then:*

- $\chi^2(G) \leq 7$ if $\Delta(G) = 3$
- $\chi^2(G) \leq \Delta(G) + 5$ if $4 \leq \Delta(G) \leq 7$
- $\chi^2(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor + 1$ if $\Delta(G) \geq 8$

This conjecture remains open, except in the case of $\Delta(G) = 3$, which was proved by Thomassen [10].

Note that any graph G satisfies $\chi^2(G) \geq \Delta(G) + 1$. Indeed, if we consider a vertex of maximal degree and its neighbors, they form a set of $\Delta(G) + 1$ vertices, any two of which are adjacent or have a common neighbor. Hence at least $\Delta(G) + 1$ colors are needed for a 2-distance coloring of G . It is therefore natural to ask when this lower bound is reached. For that purpose, we can study, as suggested by Wang and Lih [11], what conditions on the sparseness of the graph can be sufficient to ensure the equality holds. The sparseness of a graph can for example be measured by its girth, as defined below.

Definition 2. *The girth of a graph G , denoted $g(G)$, is the length of a shortest cycle.*

Conjecture 2 (Wang and Lih [11]). *For any integer $k \geq 5$, there exists an integer $M(k)$ such that for every planar graph G verifying $g(G) \geq k$ and $\Delta(G) \geq M(k)$, $\chi^2(G) = \Delta(G) + 1$.*

Conjecture 2 was proved by Borodin, Ivanova and Noestroeva [4] to be true for $k \geq 9$ and false for $k \in \{5, 6\}$, but they left the cases $k \in \{7, 8\}$ open. More precisely, they proved the following.

Theorem 1 (Borodin et al [4]). *There exist planar graphs G with $g(G) = 6$ such that $\chi^2(G) > \Delta(G) + 1$ for arbitrarily large $\Delta(G)$.*

For any planar graph G , $\chi^2(G) = \Delta(G) + 1$ in each of the following cases:

- | | |
|---|---|
| (1) $\Delta(G) \geq 3$ and $g(G) \geq 24$ | (5) $\Delta(G) \geq 7$ and $g(G) \geq 11$ |
| (2) $\Delta(G) \geq 4$ and $g(G) \geq 15$ | (6) $\Delta(G) \geq 9$ and $g(G) \geq 10$ |
| (3) $\Delta(G) \geq 5$ and $g(G) \geq 13$ | (7) $\Delta(G) \geq 16$ and $g(G) \geq 9$ |
| (4) $\Delta(G) \geq 6$ and $g(G) \geq 12$ | |

An extension of 2-distance k -coloring is the 2-distance k -list-coloring, where instead of having the same set of k colors for the whole graph, every vertex is assigned some set of k colors and has to be colored from it. Given a graph G , we call $\chi_\ell^2(G)$ the minimal integer k such that a 2-distance k -list-coloring exists. Obviously, 2-distance coloring is a subcase of 2-distance list-coloring (where the same color list is assigned to every vertex), so for any graph G , $\chi_\ell^2(G) \geq \chi^2(G)$. Kostochka and Woodall [8] even conjectured that it is actually an equality.

Conjecture 3 (Kostochka and Woodall [8]). *Any graph G verifies $\chi_\ell^2(G) = \chi^2(G)$.*

This conjecture is still open.

Borodin, Ivanova and Noestroeva [5] strengthened Theorem 1 by extending it to list-coloring, and closing the gap between 6 and 9.

Theorem 2 (Borodin et al [5]). *There exist planar graphs G with $g(G) \leq 6$ such that $\chi_\ell^2(G) > \Delta(G) + 1$ for arbitrarily large $\Delta(G)$.*

If G is a planar graph, then $\chi_\ell^2(G) = \Delta(G) + 1$ in each of the following cases:

- | | |
|---|---|
| (1) $\Delta(G) \geq 3$ and $g(G) \geq 24$ | (5) $\Delta(G) \geq 7$ and $g(G) \geq 11$ |
| (2) $\Delta(G) \geq 4$ and $g(G) \geq 15$ | (6) $\Delta(G) \geq 9$ and $g(G) \geq 10$ |
| (3) $\Delta(G) \geq 5$ and $g(G) \geq 13$ | (7) $\Delta(G) \geq 16$ and $g(G) \geq 9$ |
| (4) $\Delta(G) \geq 6$ and $g(G) \geq 12$ | (8) $\Delta(G) \geq 15$ and $g(G) = 8$ |
| | (9) $\Delta(G) \geq 30$ and $g(G) = 7$ |

Another way to measure the sparseness of a graph is through its maximum average degree as defined below. The average degree of a graph G , denoted $\text{ad}(G)$, is $\frac{\sum_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|}$.

Definition 3. *The maximum average degree of a graph G , denoted $\text{mad}(G)$, is the maximum on every subgraph H of $\text{ad}(H)$.*

Intuitively, this measures the sparseness of a graph because it states how great the concentration of edges in a same area can be. For example, stating that $\text{mad}(G)$ has to be smaller than 2 means that G cannot be anything but a tree. Dolama and Sopéna [6] used this measure of sparseness and proved the following result:

Theorem 3 (Dolama and Sopéna [6]). *Every graph with $\Delta(G) \geq 4$ and $\text{mad}(G) < \frac{16}{7}$ verifies $\chi^2(G) = \Delta(G) + 1$.*

Fortunately, we have a very handy formula, derived from Euler's formula, which links these two notions of sparseness in the case of planar graphs.

Lemma 1. *For every planar graph G , $(\text{mad}(G) - 2)(g(G) - 2) < 4$.*

Proof (Folklore). Let H be a subgraph of G such that $\text{mad}(G) = \text{ad}(H) = \frac{2|E(H)|}{|V(H)|}$. Euler's formula states that: $|E(H)| - |V(H)| + 2 = |F(H)|$, where $|F(H)|$ is the number of faces of H . But $|F(H)| \leq \frac{2|E(H)|}{g(H)}$.

$$\begin{aligned} \text{So } |E(H)| - |V(H)| &< \frac{2|E(H)|}{g(H)} \\ \frac{2|E(H)|g(H)}{|V(H)|} &< \frac{4|E(H)|}{|V(H)|} + 2g(H) \\ \frac{2|E(H)|g(H)}{|V(H)|} - \frac{4|E(H)|}{|V(H)|} - 2g(H) + 4 &< 4 \\ (\text{mad}(G) - 2)(g(H) - 2) &< 4 \end{aligned}$$

Since $g(H) \geq g(G)$, $(\text{mad}(G) - 2)(g(G) - 2) < 4$. □

As a consequence, we can transpose any theorem holding for an upper bound on $\text{mad}(G)$ into a theorem holding for a lower bound on $g(G)$ under the condition that G be planar. In particular, Theorem 3 implies that for every planar graph G with $g(G) \geq 16$ and $\Delta(G) \geq 4$, $\chi^2(G) = \Delta(G) + 1$. This lower bound on the girth is not as good as the one stated in Theorem 1.(2) by Borodin et al, and the first goal of this internship was to try to improve Theorem 3.

2.2 Our results

We managed to improve Theorem 3 into the following.

Theorem 4. *Every graph G with $\Delta(G) \geq 4$ and $\text{mad}(G) < \frac{7}{3}$ verifies $\chi^2(G) = \Delta(G) + 1$.*

Theorem 4 happens to be optimal, as Montassier [9] pointed out that there exists a graph G with $\text{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6 > \Delta(G) + 1$ (see Figure 1).

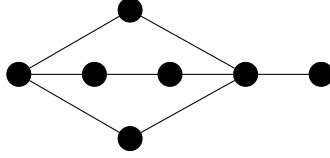


Figure 1: A graph G with $\text{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6$.

We can transpose it to planar graphs with a lower bound on the girth using Lemma 1, and it is then an improvement of Theorem 1.(2).

Corollary 1. *Every planar graph G with $\Delta(G) \geq 4$ and $g(G) \geq 14$ verifies $\chi^2(G) = \Delta(G) + 1$.*

However, it is not comparable to the more general result in Theorem 2 since we are not considering list-coloring.

This being done, we wanted to know whether we could improve similarly the other cases of Theorem 2. More precisely, we wanted to find a function f and a constant M such that any graph G with $\text{mad}(G) < M$ and $\Delta(G) \geq f(M)$ would verify $\chi^2(G) = \Delta(G) + 1$: M being as big as possible. We obtained the following result, which shows that M can be arbitrarily close to $\frac{8}{3}$.

Theorem 5. *For every $\epsilon > 0$, any graph G with $\text{mad}(G) < \frac{8}{3} - \epsilon$ and $\Delta(G) \geq \max(5, \frac{4}{3\epsilon})$ verifies $\chi^2(G) = \Delta(G) + 1$.*

We can, once again, transpose it to planar graphs with a lower bound on the girth using Lemma 1, and it is then an improvement of the cases (3) to (7) of Theorem 2.

Corollary 2. *For any planar graph G , $\chi^2(G) = \Delta(G) + 1$ in each of the following cases:*

- $\Delta(G) \geq 5$ and $g(G) \geq 12$

- $\Delta(G) \geq 6$ and $g(G) \geq 11$
- $\Delta(G) \geq 8$ and $g(G) \geq 10$
- $\Delta(G) \geq 15$ and $g(G) \geq 9$

We know that $M < 3$ due to the family of graphs presented in Figure 2.

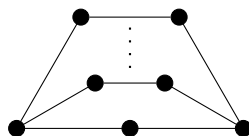


Figure 2: A graph G_p with $\text{mad}(G_p) = 3 - \frac{5}{2p+1}$, $\Delta(G_p) = p$ and $\chi^2(G_p) = \Delta(G_p) + 2$.

We do not know yet what happens between $\frac{8}{3}$ and 3. Is the planarity hypothesis necessary to provide lower bounds on $\Delta(G)$ such that $\text{mad}(G)$ can be arbitrarily close to 3? It seems anyway that the method we used (see Section 3) cannot be extended to fill the gap. We also know that if there exists a family of graphs G_p of increasing maximal degree, where $\text{mad}(G_p)$ tends to $\frac{8}{3}$ and $\chi_\ell^2(G) > \Delta(G_p) + 1$, then it is not planar, and has a huge number of vertices compared to its maximal degree.

More generally, we wondered if we could get similar results when adding a constant number of colors, ie if we could find a function h and a constant N such that for any graph G such that $\text{mad}(G) < N$ would verify $\chi_\ell^2(G) \leq \Delta(G) + h(N)$: N being as big as possible. We obtained the following result, which shows that N can be arbitrarily close to 4.

Theorem 6. *Every graph G with $\text{mad}(G) < 4 - \epsilon$ verifies $\chi_\ell^2(G) \leq \Delta(G) + h(\epsilon)$, where $h(\epsilon) \sim \frac{40}{\epsilon}$ when $\epsilon \rightarrow 0$.*

The bound on N cannot be improved; the family of graphs presented in Figure 3 proves that $N < 4$.

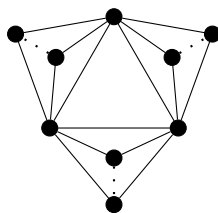


Figure 3: A graph G_p with $\text{mad}(G_p) = 4 - \frac{4}{p}$, $\Delta(G_p) = p$ and $\chi^2(G) = \frac{3\Delta(G_p)}{2}$.

The existence of this family also justifies that $h(\epsilon)$ cannot be smaller than $\frac{2}{\epsilon}$. We do not know whether this lower bound is actually the optimal value, but we are aware of the fact that we could probably get a more refined function h .

Those are the main results we were able to get about the 2-distance coloring of sparse graphs (others involved an improvement of Theorem 1.(1), whose proof is disproportionately long, and more precise results for $\chi_\ell^2(G) \leq \Delta(G) + 2$ and $\Delta(G) + 3$, which are not quite relevant yet). We will now explain the principle of a discharging method and sketch the proofs of the above-cited results.

3 Discharging methods

3.1 A brief introduction

The discharging method was introduced in the early 20st century, and is essentially known for being used by Appel and Haken in 1976 in order to prove the 4-color theorem.

Theorem 7 (Appel et al [1, 2]). *Every planar graph can be colored with 4 colors.*

It can be relevant to point out that so far, every proof of Theorem 7 uses the discharging method.

Very roughly, here are the main steps of a discharging proof. The forbidden configurations are facultative.

- Have a conjecture.
- Consider a minimal counter-example G to this conjecture (minimal according to some order, for example in terms of number of vertices, or in terms of subgraph).
 - Prove there are a few configurations G cannot contain. A configuration is a pattern in the graph, like two vertices of degree 2 that are adjacent, or a clique of size 4. This means assume by contradiction G contains a given configuration, apply the minimality hypothesis to solve the problem on a smaller graph ("smaller" according to the same order that was used for the definition of "minimal") and derive from this solution a solution to the problem on G .
 - Attribute weights to elements of G (vertices, edges, faces, incidences...).
 - Design discharging rules (under which conditions weight transfers can take place).
 - Apply them to G .
 - Use the forbidden configurations and weight conservation to derive a contradiction from the final weights in G (traditionally, G is planar and the contradiction is derived from Euler's formula, but this is only one out of many possibilities).
- Hence no counter-example exists.

This is the order in which the proofs are presented, but it is of course not the order in which the proofs are found. In practice, it is more of a constant readjustment, by trial and error.

To get a more practical notion of how we used this discharging method, let us consider a very simple example. Assume we want to study what conditions on a graph G are sufficient to ensure $\chi_l^2(G) \leq \Delta(G) + 2$. Let us look more precisely for an upper bound on $\text{mad}(G)$.

- The conjecture is: "Any graph G such that $\text{mad}(G) < a$ verifies $\chi_l^2(G) \leq \Delta(G) + 2$ ".
- What weights could we attribute in order to get a contradiction at the end?
 - Knowing that we have an upper bound on $\text{mad}(G)$, we could attribute to each vertex its degree as a weight, then try to prove that at the end, every vertex has a weight of at least a , hence the contradiction ($\frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|} \leq \text{mad}(G) < a$, and by conservation of weight, if every vertex has at the end a weight of at least a , then $\sum d(v) \geq a \times |V|$).
- What configurations can we forbid?
 - We can for example forbid that there be a vertex of degree 1. Indeed, assume that in a minimal counter-example G (minimal meaning that all its proper subgraphs verify the property), there exists a vertex v of degree 1. We use the minimality of G to 2-distance color $G \setminus \{v\}$ with at most $\Delta(G) + 2$ colors. Then, there are at most $\Delta(G)$ vertices that are adjacent to or have a common neighbor with v . Consequently, there are two free colors for v , and there exists a 2-distance $(\Delta(G) + 2)$ -coloring of G .
 - We can also forbid that there is a path $x-u-v-w-y$ where u, v, w are of degree 2. Assume G contains such a path. We use the minimality of G to color $G \setminus \{v, w\}$. Vertex w has at most $|N(y) \setminus \{w\} \cup \{y, u\}| \leq \Delta(G) + 1$ constraints, so there is a free color for w (the number of constraints of w is the number of different colors used on vertices that are adjacent to or have a common neighbour with w). Vertex v has at most 4 constraints, hence there is a free color for v if we assume the size of the color lists is at least 5.

- The conjecture is now: "Any graph G such that $\text{mad}(G) < a$ verifies $\chi_\ell^2(G) \leq \max(\Delta(G) + 2, 5)$ ".
- What discharging rules can we design so that every vertex get a weight of at least a at the end?
 - According to our forbidden configurations, there is no vertex of degree 1, and every vertex of degree 2 has a neighbor that is of degree at least 3. Consequently, all we have to ensure is that every vertex of degree at least 3 can afford to give $a - 2$ to all its neighbors and still have a weight of at least a after that. In other words, we only need $3 - a \geq 3(a - 2)$, ie $a \leq \frac{9}{4}$.
- The conjecture is now: "Any graph G such that $\text{mad}(G) < \frac{9}{4}$ verifies $\chi_\ell^2(G) \leq \max(\Delta(G) + 2, 5)$ ".
- We have a single discharging rule to design : every vertex of degree at least 3 gives $\frac{1}{4}$ to each of its neighbors of degree 2.
- Qed. Any graph G such that $\Delta(G) \geq 3$ and $\text{mad}(G) < \frac{9}{4}$ verifies $\chi_\ell^2(G) \leq \Delta(G) + 2$.

This is obviously far from being an optimal result, but it is a simple illustration of the discharging method.

Another interest of the discharging method, beside the fact that it is a powerful tool to prove things about the structure of graphs, is that a polynomial algorithm can generally be derived from a discharging proof. Indeed, we actually prove that every graph verifying the hypotheses of the theorem contains one of the forbidden configurations. Since the hypotheses are hereditarily verified, at each step, we spot a configuration, remove the corresponding vertices, color inductively the rest of the graph, and extend the coloring to the removed vertices. Besides, spotting a configuration can be sped up by applying the discharging rules: near those configurations, some weights do not have the expected value (in our case, a vertex will have a weight smaller than the upper bound on the maximum average degree). Usually, it is even a small polynomial.

3.2 Using local discharging

We say a discharging proof is local when the configurations and the discharging rules are all of finite size. Historically, all discharging proofs are local, and the notion of global discharging (See Section 3.3) was only introduced in 2005, by Borodin, Ivanova and Kostochka [3]. Only one of our proofs is local.

We say a graph G is *minimal* for a property when G verifies it but none of its proper subgraphs does.

Proof of Theorem 6

Let $2 > \epsilon > 0$.

We call $h(\epsilon) = \max(5M - 6, 2M + 3)$, where M is defined as the solution to $M - (4 - \epsilon) = M \times (1 - \frac{\epsilon}{2})$, ie $M = \frac{8}{\epsilon} - 2$. (So $h(\epsilon) = \max(\frac{40}{\epsilon} - 16, \frac{16}{\epsilon} - 1)$.)

We want to prove that every graph G with $\text{mad}(G) < 4 - \epsilon$ admits a 2-distance $(\Delta(G) + h(\epsilon))$ -list-coloring. We do that by a discharging method.

Assume the above is false. Take G a minimal graph with $\text{mad}(G) < 4 - \epsilon$ that does not admit a 2-distance $(\Delta(G) + h(\epsilon))$ -list-coloring. We can assume without loss of generality that G is connected.

We call *weak* a vertex of degree 2 or 3 that has at most one neighbor of degree M^+ .

In the figures, we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label ' i ' means "exactly i neighbors",

the label ' i^+ ' (resp. ' i^- ') means that it has at least (resp. at most) i neighbors. The label ' w ' means the vertex is weak. Note that the white vertices may coincide with other vertices.

We first prove that G cannot contain any of the configurations (C_1) and (C_2) depicted in Figure 4. Configuration (C_1) is a vertex u of degree 1. Configuration (C_2) is a vertex u of degree M^- that has a weak neighbor x , and at most 3 neighbors of degree 4^+ , among which at most one is of degree M^+ .

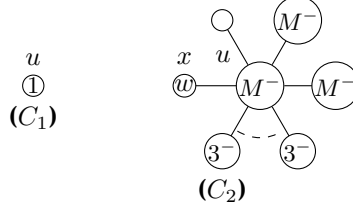


Figure 4: Forbidden configurations for Theorem 6.

Lemma 2. G cannot contain Configurations (C_1) nor (C_2) .

Proof. (C_1) We color $G \setminus \{u\}$ using the minimality of G . Vertex u has at most $\Delta(G)$ constraints, so there is a free color for u , a contradiction.

(C_2) We remove the (u, x) edge, and use the minimality of G to color the resulting graph. We discolor u and x , restore the edge (u, x) and extend the coloring as follows: u has at most $\Delta(G) + 2M + \max(3(M - 3), 0) + 2$ constraints, so there is a free color for u , and x has at most $\Delta(G) + M + M$ constraints, so there is a free color for x , a contradiction. □

We then attribute to each vertex a weight equal to its degree. We design two discharging rules R_1, R_2 and prove that each vertex of G enjoys a weight of at least $4 - \epsilon$ at the end of the discharging.

Discharging rule R_1 (see Figure 5) states that a vertex of degree at least M gives $1 - \frac{\epsilon}{2}$ to each of its neighbors. Discharging rule R_2 states that a vertex of degree less than M gives $1 - \frac{\epsilon}{2}$ to each of its weak neighbors.



Figure 5: Discharging rules R_1, R_2 for Theorem 6.

Lemma 3. Each vertex of G has a weight of at least $4 - \epsilon$ after application of the discharging rules.

Proof. Let u be a vertex of G . Thanks to Configuration (C_1) , we have $d(u) \geq 2$.

- u gives some weight away.
 - If $d(u) \geq M$, (R_1) is applied, and by definition of M , vertex u gives $1 - \frac{\epsilon}{2}$ to each of its neighbors and still has a weight of at least $4 - \epsilon$.
 - If $d(u) < M$, (R_2) is applied and u has a weak neighbor x . Since (C_2) is forbidden, u is in one of these two situations:
 - * u has at least two neighbors of degree M^+ . According to R_1 , they each give $1 - \frac{\epsilon}{2}$ to u . Then u has at most $d - 2$ weak neighbors, and $d - (4 - \epsilon) + 2(1 - \frac{\epsilon}{2}) \geq (d - 2)(1 - \frac{\epsilon}{2})$, so u has a weight of at least $4 - \epsilon$ after application of the discharging rules.

* u has at least four neighbors of degree 4^+ . So u has at most $d - 4$ weak neighbors, and $d - (4 - \epsilon) \geq (d - 4)(1 - \frac{\epsilon}{2})$, hence u has a weight of at least $4 - \epsilon$ after application of the discharging rules.

- u gives no weight away.
 - $d(u) \geq 4$. Then u still has a weight of at least $4 - \epsilon$ after application of the discharging rules.
 - u is a weak vertex. Then, according to (C_2) , it can't be adjacent to another weak vertex, so it gives nothing away and receives $1 - \frac{\epsilon}{2}$ from each of its neighbors. After application of the discharging rules, it has a weight at least $2 + 2 \times (1 - \frac{\epsilon}{2}) = 4 - \epsilon$
 - $d(u) \leq 3$ and u is not weak. Then, u has at least two neighbors of degree at least M , so u receives at least $2 \times (1 - \frac{\epsilon}{2})$. It had initially a weight of at least 2 and gave nothing away, meaning that it has a weight of at least $4 - \epsilon$ after application of the discharging rules.

As a conclusion, every vertex in G enjoys a weight of at least $4 - \epsilon$ after application of the discharging rules. \square

Since the weight is conserved, this means $\sum d(v) \geq (4 - \epsilon) \times |V|$, hence $4 - \epsilon \leq \frac{\sum d(v)}{|V|} = \frac{2|E|}{|V|} \leq \text{mad}(G) < 4 - \epsilon$, a contradiction.

3.3 Using semi-local discharging

As mentioned earlier, the notion of global discharging was introduced by Borodin, Ivanova and Kostochka [3]. A discharging method is *global* when we consider arbitrarily large configurations and make some weight travel arbitrarily far along those configurations. We use a sort of semi-local discharging, where the weight always stays in a finite area, but where both the forbidden configurations and the discharging rules take into account structures of arbitrarily large size. We have two proofs of this kind: Proofs of Theorems 4 and 5, but we will only sketch the main ideas and leave the technical parts to the Appendix.

Proof of Theorem 4

We want to prove that every graph G with $\text{mad}(G) < \frac{7}{3}$ admits a 2-distance $(\max(5, \Delta(G) + 1))$ -coloring. We do that by a discharging method, whose main steps we present here (see Section A for the details).

Assume the above is false. Take G a minimal graph with $\text{mad}(G) < \frac{7}{3}$ that does not admit a 2-distance $(\max(5, \Delta(G) + 1))$ -coloring.

In the figures of this proof, a label $T(v, a_4)$ inside a vertex means that $T(v, a_4)$ exists, as defined below.

A configuration $T(v, a_4)$ (see Figure 6) is inductively defined as a vertex v of degree 4 with neighbors a_1, a_2, a_3, a_4 , where for $i \in \{1, 2, 3\}$, vertex v is 2-linked by a path $v-a_i-b_i-w_i$ either to a vertex w_i of degree at most 3 or to a configuration $T(w_i, b_i)$.

Now we define configurations (C_1) to (C_5) (see Figure 7).

- (C_1) is a vertex of degree 0 or 1.
- (C_2) is a vertex 3-linked to a vertex not of maximal degree.
- (C_3) is a vertex of degree 3 that is 2-linked to two vertices of degree 3, and 1-linked to a vertex of degree at most 3.
- (C_4) is a vertex u of degree at most 3 that is 2-linked by a path $u-y-x-v$ to a vertex v such that $T(v, x)$ exists.

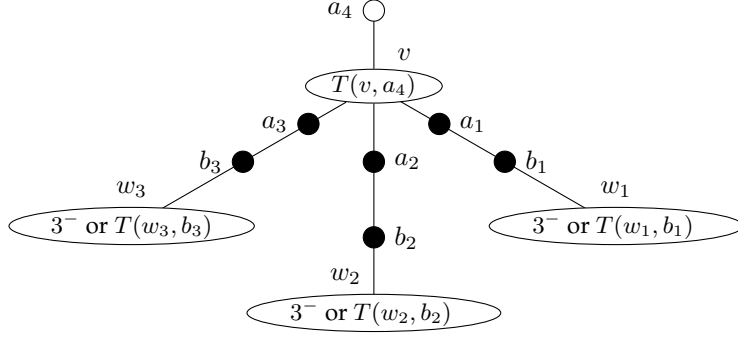


Figure 6: A $T(v, u_4)$.

- (C_5) is a vertex u of degree 3 that is 2-linked to two vertices, and 1-linked by a path $u-x-v$ to a vertex v such that $T(v, x)$ exists.

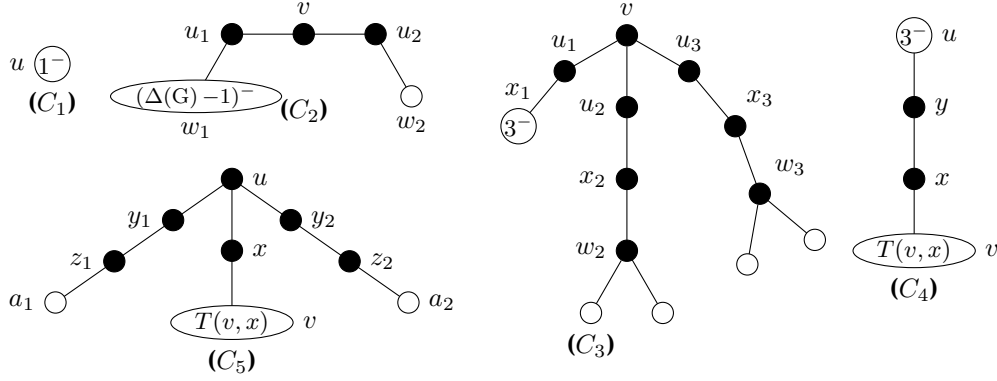


Figure 7: Forbidden configurations for Theorem 4.

Lemma 4. *The graph G does not contain any of Configurations (C_1) to (C_5) .*

The following lemma will ensure that the discharging rules we introduce later are well-defined.

Lemma 5. *In a graph H where (C_4) is forbidden, and x and y are two vertices of degree 4 such that a path $x-a-b-y$ (with a and b of degree 2) exists, $T(x, a)$ and $T(y, b)$ cannot both exist.*

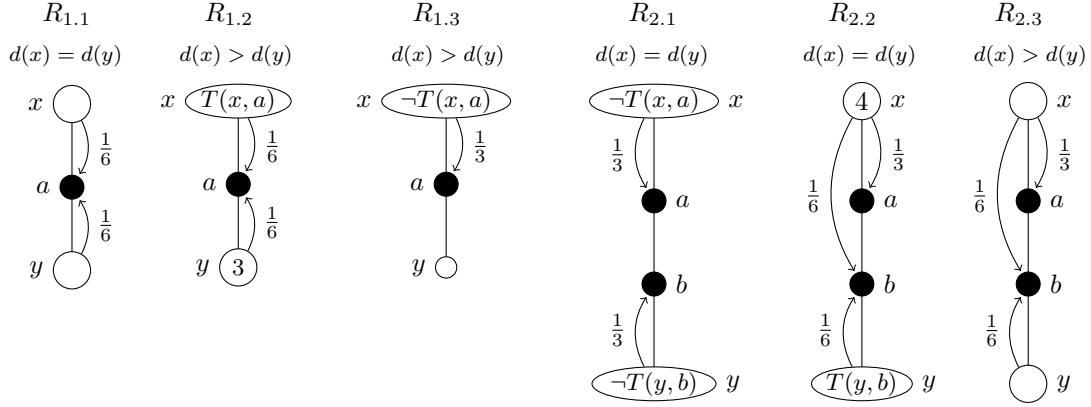
We design discharging rules R_1, R_2, R_3 (see Figure 8). We use them in the proof of Lemma 6, where the initial weight of a vertex equals its degree, and its final weight is shown to be at least $\frac{7}{3}$. For any two vertices x and y of degree at least 3, with $d(x) \geq d(y)$,

- Rule R_1 is when x and y are 1-linked by a path $x-a-y$.
 - $(R_{1.1})$ If $d(x) = d(y)$, then both x and y give $\frac{1}{6}$ to a .
 - $(R_{1.2})$ If $d(x) > d(y)$ and $T(x, a)$ exists, then both x and y give $\frac{1}{6}$ to a .
 - $(R_{1.3})$ If $d(x) > d(y)$ and $T(x, a)$ does not exist, then x gives $\frac{1}{3}$ to a .
- Rule R_2 is when x and y are 2-linked by a path $x-a-b-y$.
 - $(R_{2.1})$ If $d(x) = d(y)$ and neither $T(x, a)$ nor $T(y, b)$ exist, then x (resp. y) gives $\frac{1}{3}$ to a (resp. b).

- ($R_{2.2}$) If $d(x) = d(y)$ and $T(y, b)$ exists, then x gives $\frac{1}{3}$ to a and both x and y give $\frac{1}{6}$ to b .
- ($R_{2.3}$) If $d(x) > d(y)$, then x gives $\frac{1}{3}$ to a and both x and y give $\frac{1}{6}$ to b .
- Rule R_3 is when x and y , both of degree at least 4, are 3-linked by a path $x - a - b - c - y$. Then x gives $\frac{1}{3}$ to a and $\frac{1}{6}$ to b , and symmetrically for y .

Rule 1: x and y are 1-linked

Rule 2: x and y are 2-linked



Rule 3: x and y are 3-linked.

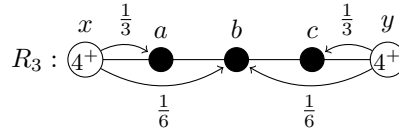


Figure 8: Discharging rules R_1, R_2, R_3 for Theorem 4.

We use these discharging rules to prove the following lemma:

Lemma 6. A graph H that does not contain Configurations (C_1) to (C_5) verifies $\text{mad}(G) \geq \frac{7}{3}$.

Lemmas 4 and 6 imply that $\text{mad}(G) \geq \frac{7}{3}$, a contradiction. Hence the theorem holds. \square

The limitation in transposing the above proof to list-coloring lies in Configuration (C_4) : while proving this configuration is forbidden, we often affect the same colour to different vertices in order to complete the coloring, which is hard to transpose to list-coloring as the color lists can differ.

Also, the bound on $\frac{7}{3}$ is tight (see Figure 1), and in our proof, the limitation of $\frac{7}{3}$ lies in the structure of $T(v, u_4)$: if we aim at an upper bound of $2 + a$ on the maximum average degree, v has to 'feed' $\frac{3}{2}a$ to all its branches except u_4 , and to be able to give $\frac{1}{2}a$ to u_4 if it is part of a bigger $T(w, x_4)$, so that the induction can be applied. Hence, we need $4 - (2 + a) \geq 3 * (\frac{3}{2}a) + \frac{1}{2}a$, ie $2 + a \leq \frac{7}{3}$.

Proof of Theorem 5

Let k be a constant integer, $k > 4$.

In the figures of this proof, the label ' $T(v, a)$ ' inside a vertex v means that $T(v, a)$ exists, as defined below. The structure is quite close to the one defined in Proof of Theorem 4, though it is not a generalization of it.

A configuration $T(v, a_1)$ (see Figure 9), is inductively defined as a vertex v of degree k with neighbors a_1, a_2, \dots, a_k , where for $i \in \{2, \dots, k\}$, vertex v is 2-linked by a path $v-a_i-b_i-w_i$ either to a vertex w_i of degree at most $k-2$ or to a configuration $T(w_i, b_i)$.

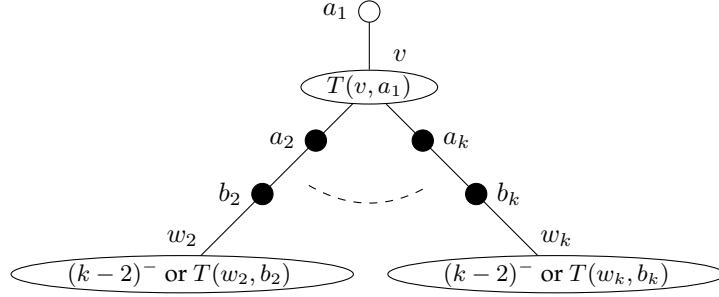


Figure 9: $T(v, a_1)$.

We define configurations (C_1) to (C_6) (see Figure 10).

- (C_1) is a vertex of degree 0 or 1.
- (C_2) is some vertex 3-linked to a vertex not of maximal degree.
- (C_3) is a vertex of degree at most $k-1$ that is 2-linked to a vertex of degree at most $k-2$.
- (C_4) is a vertex w_2 of degree at most $k-1$ that is 2-linked (by a path $w_2 - u_2 - u_1 - w_1$) to a vertex w_1 such that $T(w_1, u_1)$ exists.
- (C_5) is a vertex v of degree 3 that is 1-linked to two vertices, and 1-linked (by a path $v - u_1 - w_1$) to a vertex w_1 that is of degree at most 4 or such that $T(w_1, u_1)$ exists.
- (C_6) is a vertex v of degree 3 that is 1-linked (by two paths $v - u_2 - w_2, v - u_3 - w_3$) to two vertices w_2, w_3 , where w_2 is of degree at most 7 or such that $T(w_2, u_2)$ exists, and whose third neighbor is a vertex w_1 that is of degree at most 3.

In the following lemma, we actually use k instead of $\Delta(G)$ in order to ensure that any subgraph of G admits a $(k+1)$ -list-coloring even though $\Delta(G)$ can decrease.

Lemma 7. *If G is a minimal graph such that $\Delta(G) \leq k$ and G admits no 2-distance $(k+1)$ -list-coloring, then G cannot contain any of Configurations (C_1) to (C_4) , nor (C_5) if $k \geq 6$, nor (C_6) if $k \geq 9$.*

The following lemma will ensure that the discharging rules we introduce later are well-defined.

Lemma 8. *In a graph G where (C_4) is forbidden, and x and y are two vertices of degree $\Delta(G)$ that are 2-linked through a $-b$ (a being the closest to x), $T(x, a)$ and $T(y, b)$ cannot both exist.*

We introduce $\epsilon = \frac{4}{3k}$, $M_1 = \frac{8-3\epsilon}{2+3\frac{\epsilon}{5}}$, which verifies $\forall n \geq M_1, n - (\frac{8}{3} - \epsilon) \geq n(\frac{1}{3} - \frac{\epsilon}{2})$, and $M_2 = \frac{8-3\epsilon}{1+3\epsilon}$, which verifies $\forall n \geq M_2, n - (\frac{8}{3} - \epsilon) \geq n(\frac{2}{3} - \epsilon)$. Note that $M_1 < 4$, $M_2 < 8$.

We design discharging rules R_1, R_2, R_3 and R_4 (see Figure 11). We use them in the proof of Lemma 9, where the initial weight of a vertex equals its degree, and its final weight is shown to be at least $\frac{8}{3} - \epsilon$. For any vertex x of degree at least 3,

- Rule R_1 : if $3 \leq d(x) < M_1$, then x gives $\frac{1}{3} - \frac{\epsilon}{2}$ to every neighbor a of degree 2 that is adjacent to a vertex y of degree less than M_2 or such that $T(y, a)$ exists.
- Rule R_2 : if $M_1 \leq d(x) < M_2$, then x gives $\frac{1}{3} - \frac{\epsilon}{2}$ to each of its neighbors.

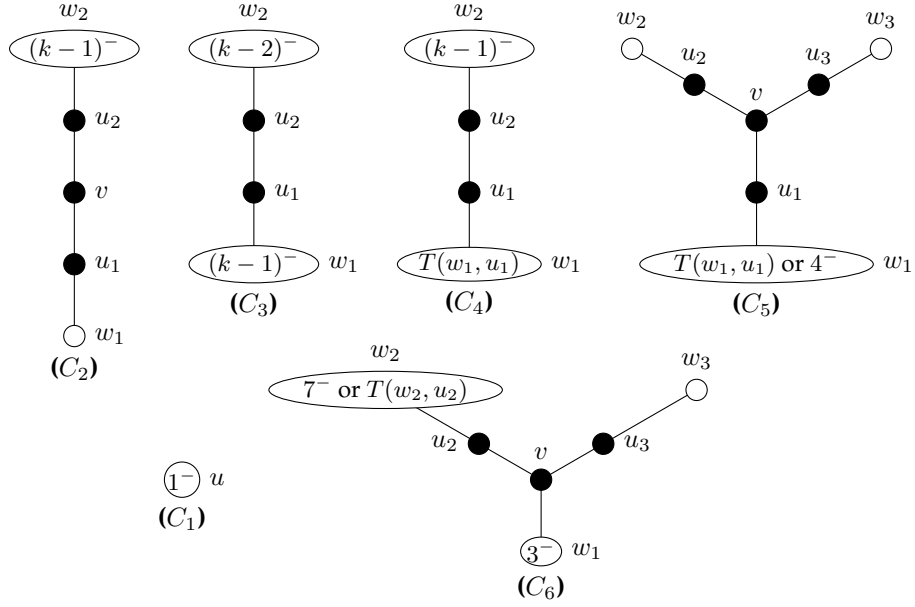
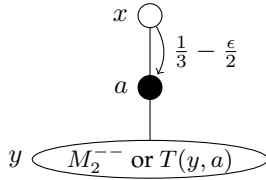


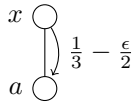
Figure 10: Forbidden configurations for Theorem 5.

- Rule R_3 : if $M_2 \leq d(x) < k$, then x gives $\frac{2}{3} - \epsilon$ to each of its neighbors.
- Rule R_4 : if $d(x) = k$, then for a neighbor a of x , x gives $\frac{2}{3} - \epsilon$ to a if $T(x, a)$ does not exist ($R_4(i)$), $\frac{1}{3} - \frac{\epsilon}{2}$ if it does ($R_4(i')$). For a vertex y of degree 2 that is 1-linked (by a path $y - b - a - x$) to x , if $T(x, a)$ does not exist and either $T(y, b)$ does or $d(y) \leq k - 2$, then x gives $\frac{1}{3} - \frac{\epsilon}{2}$ to b ($R_4(ii)$).

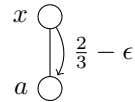
Rule 1: $3 \leq d(x) < M_1$



Rule 2: $M_1 \leq d(x) < M_2$



Rule 3: $M_2 \leq d(x) < k$



Rule 4: $d(x) = k$

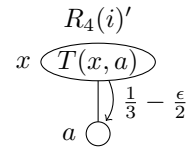
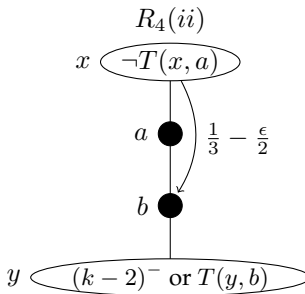
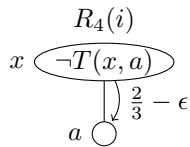


Figure 11: Discharging rules R_1, R_2, R_3 and R_4 for Theorem 5.

We use these discharging rules to prove the following lemma:

Lemma 9. *A graph G with $\Delta(G) \leq k$ that does not contain Configurations (C_1) to (C_4) , nor (C_5) if $k \geq 6$, nor (C_6) if $k \geq 9$ verifies $\text{mad}(G) \geq \frac{8}{3} - \epsilon$.*

We prove Theorem 5 by contradiction. Let G be a minimal graph with $\Delta(G) \leq k$ and $\text{mad}(G) < \frac{8}{3} - \frac{4}{3k}$ such that G does not admit a $(k+1)$ -list-coloring. Graph G is also a minimal graph such that $\Delta(G) \leq k$ and G does not admit a $(k+1)$ -list-coloring (all its proper subgraphs verify $\Delta \leq k$ and $\text{mad} < \frac{8}{3} - \frac{4}{3k}$, so they admit a $(k+1)$ -list-coloring). By Lemma 7, graph G cannot contain (C_1) to (C_4) (nor (C_5) if $k \geq 6$, nor (C_6) if $k \geq 9$). Lemma 9 implies that $\text{mad}(G) \geq \frac{8}{3} - \epsilon$, a contradiction. \square

Just as in the proof of Theorem 4, the limitation on the upper bound of the maximum average degree lies in the structure of the $T(v, a_1)$. This does not say that our proof is optimal, all we can claim is that this proof cannot be trivially modified to improve the bounds (both on $\Delta(G)$ and on the maximum average degree).

4 Future work

As we pointed out all along this report, there are a few questions that emerge naturally from this internship and should be studied in order to make this work as coherent as possible. Can we extend Theorem 4 to list-coloring? Is the upper bound of $\frac{8}{3}$ optimal in Theorem 5? What is the optimal value of $h(\epsilon)$ in Theorem 6?

There are also a few less direct questions: what about 2-distance $\mathcal{O}(\Delta(G))$ -coloring? It is known that for any graph G , $(2 \text{mad}(G) \Delta(G))$ colors are enough: can this bound be improved? But the deepest questions are about the method, not about the results. What else can we do with this global (or semi-global) discharging method? Is there some general approach for this? Are there results that can be obtained with a global discharging method, but not with a semi-global discharging method? With a semi-global discharging method, but not with a local discharging method?

Addendum

Ivanova published online on May, 30th 2011 a paper [7] that improves Corollary 2 for $\Delta(G) > 5$ and matches it for $\Delta(G) = 5$. Since these results are restricted to planar graphs, this does not affect Theorem 5.

Theorem 8 (Ivanova [7]). *If G is a planar graph, then $\chi_\ell^2(G) = \Delta(G) + 1$ in each of the following cases:*

- (1) $\Delta(G) = 5$ and $g(G) \geq 12$
- (2) $\Delta(G) \geq 6$ and $11 \geq g(G) \geq 10$
- (3) $\Delta(G) \geq 10$ and $8 \leq g(G) \leq 9$
- (4) $\Delta(G) \geq 16$ and $g(G) = 7$

Acknowledgements

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Appendix

A Proof of Theorem 4

A.1 Proof of Lemma 4

Proof. We prove Lemma 4 by assuming G contains one of the configurations (C_1) to (C_5) , using the minimality of G to color one of its subgraphs, and extending the coloring to the whole graph, hence obtaining a contradiction. In the following proofs, a *constraint* of a vertex u is an already colored vertex that is adjacent to or has a common neighbor with u . Two constraints with the same color count as one.

We follow the notations introduced on Figure 7.

Claim 1. G cannot contain (C_1) .

Proof. Using the minimality of G , we color $G \setminus \{u\}$. Since $d(u) \leq 1$, vertex u has at most $\Delta(G)$ constraints. There are at least $\Delta(G) + 1$ colors, so the coloring of $G \setminus \{u\}$ can be extended to G . \diamond

Claim 2. G cannot contain (C_2) .

Proof. Using the minimality of G , we color $G \setminus \{v, u_1\}$. Vertex u_1 has at most $|\{w_1, u_2\}| + d(w_1) - 1 = d(w_1) + 1 \leq \Delta(G)$ constraints. Hence we can color u_1 . Then v has at most 4 constraints, so we can extend the coloring of $G \setminus \{v, u_1\}$ to G . \diamond

Claim 3. G cannot contain (C_3) .

Proof. Using the minimality of G , we color $G \setminus \{v, u_1, u_2, u_3, x_2, x_3\}$. We color x_2 in a different color of that of w_3 (this is possible since it has at most 3 constraints), u_3 with the same color as x_2 (the only constraint is w_3 , and it verifies it), u_1 (at most 4 constraints), x_3 (at most 4 constraints), v (at most 4 constraints), u_2 (at most 4 constraints). Thus, we can extend the coloring of $G \setminus \{v, u_1, u_2, u_3, x_2, x_3\}$ to G . \diamond

Given a vertex u , the *neighborhood* $N(u)$ is the set of vertices that are adjacent to u .

We define a "branch" of v as a p -link from v to another vertex.

Claim 4. Let H be a graph that contains some $T(v, u_4)$. For any 2-distance 5-coloring α of $H \setminus (T(v, u_4) \cup N(u_4))$ such that two vertices that have a common neighbor in H have distinct colors, the coloring α can be extended to $(H \setminus N(u_4)) \cup \{v\}$.

In the description of a coloring procedure, we note " $x \leftarrow c$ " as a shortcut for "We assign color c to x ", " $u_4 \xleftarrow{T} c$ " as a shortcut for "We color u_4 in c and apply Claim 4 to color $T(v, u_4)$ ", and we note also " x " as a shortcut for "We color arbitrarily x with one of the colors available".

Proof. We name $v - u_i - x_i - w_i$ the vertices along a branch (u_i and x_i are of degree 2), and b_i and c_i the two other neighbors of w_i (if $d(w_i) = 3$) (see Figure 12). All along this proof, a denotes $\alpha(u_4)$.

Since (C_2) is forbidden, there is no (i, j) such that $w_i = x_j$ and $w_j = x_i$, nor such that $w_i = w_j$. However, there can be a couple (i, j) such that $w_i = u_j$, $w_j = u_i$, $x_i = x_j$. In that case, we pretend they are distinct (we assign arbitrary colors to the virtual w_i and w_j and their alleged other neighbors b_i, c_i, b_j, c_j), apply the procedure described below, get a coloring α of the resulting graph, then derive from it a coloring of the initial graph by matching α on every common vertex except x_i and then coloring x_i in one of the available colors (indeed $\alpha(u_i) \neq \alpha(u_j)$ as v is a common neighbor, and x_i has exactly three vertices at distance 2 or less). Therefore we can assume without loss of generality that no vertex u_i, x_i superposes with another. The case $w_i = w_j$ is not a problem in the following procedure (note that $w_i = w_j$ can only happen if $d(w_i) \leq 3$).

1. Vertex v is the only vertex of degree 4 in $T(v, u_4)$.

We deal with the worst-case situation, i.e. the three branches from v end with a vertex of degree 3 (if we can extend the coloring in that case, then we would be able to do the same if one or more were of degree only 2), and $w_i, b_i, c_i \in G$.

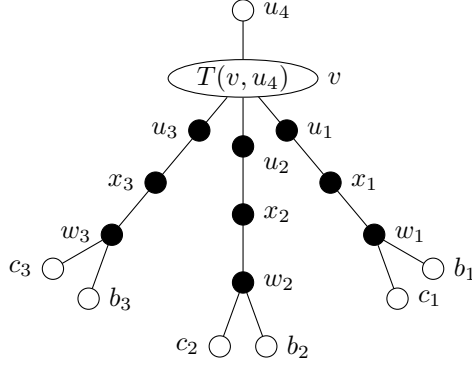


Figure 12: v is the only vertex of degree 4 in $T(v, u_4)$.

So, α is not defined on v, u_i, x_i and $T(w_i, x_i)$ if it exists, for $i \in \{1, 2, 3\}$. Since we have only 5 colors, we are always in one of the following five cases (up to permutations):

- . $\alpha(w_1) = \alpha(w_2) = \alpha(w_3), \alpha(b_1) = \alpha(b_2)$.
 - $a = \alpha(w_1)$. Then apply: $v \leftarrow \alpha(b_1), x_3, u_3, u_2, u_1, x_2, x_1$.
 - $a \neq \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_1), u_1, u_2, u_3, x_1, x_2, x_3$.
- . $\alpha(w_1) = \alpha(w_2) \neq \alpha(w_3), \alpha(b_1) = \alpha(w_3)$.
 - $a = \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_3), x_2, u_2, u_1, x_1, u_3, x_3$.
 - $a \neq \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_1), x_3, u_3, u_2, u_1, x_2, x_1$.
- . $\alpha(w_1) = \alpha(w_2) \neq \alpha(w_3), \alpha(b_1) = \alpha(b_2)$ (and we are not in the previous case).
 - $a = \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_3), u_2 \leftarrow \alpha(b_2), u_1 \leftarrow \alpha(c_1), x_1, x_2, u_3, x_3$.
 - $a \neq \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_1), x_3, u_3, u_2, u_1, x_2, x_1$.
- . $\alpha(w_1), \alpha(w_2)$ and $\alpha(w_3)$ are pairwise different, $\alpha(w_1) = \alpha(b_2)$.
 - $a = \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_3), x_2, u_2, x_1, u_1, u_3, x_3$.
 - $a \neq \alpha(w_1)$. Then apply: $v \leftarrow \alpha(w_1), x_3, u_3, u_2, x_2, u_1, x_1$.
- . $\alpha(w_1), \alpha(w_2)$ and $\alpha(w_3)$ are pairwise different, $\alpha(b_1) = \alpha(b_2) = \alpha(b_3), \alpha(c_1) = \alpha(c_2) = \alpha(c_3)$.
 - $a = \alpha(w_1)$ (up to permutation). Then apply: $v \leftarrow \alpha(b_1), u_2 \leftarrow \alpha(w_3), u_3 \leftarrow \alpha(w_2), x_2, x_3, u_1, x_1$.
 - $a = \alpha(b_1)$. Then apply: $v \leftarrow \alpha(c_1), u_1 \leftarrow \alpha(w_2), u_2 \leftarrow \alpha(w_3), u_3 \leftarrow \alpha(w_1), x_1, x_2, x_3$.
 - $a = \alpha(c_1)$. Then apply: $v \leftarrow \alpha(b_1), u_1 \leftarrow \alpha(w_2), u_2 \leftarrow \alpha(w_3), u_3 \leftarrow \alpha(w_1), x_1, x_2, x_3$.

2. Vertex v is 2-linked to exactly one vertex w_3 of degree 4 in $T(v, u_4)$ (see Figure 13).

Again, we deal with the worst-case situation. So, in this drawing, α is defined only on the w_i, b_i and c_i , for $i = 1$ or 2 . Again, because there are only 5 colors, we are in one of the following three cases;

- . $\alpha(w_1) = \alpha(w_2)$.
 - $a = \alpha(w_1)$. Then apply: $x_3 \stackrel{T}{\leftarrow} a, v \leftarrow \alpha(b_1), u_3, x_2, u_2, u_1, x_1$.
 - $a \neq \alpha(w_1)$. Then apply: $x_3 \stackrel{T}{\leftarrow} a, v \leftarrow \alpha(w_1), u_3, u_1, u_2, x_1, x_2$.

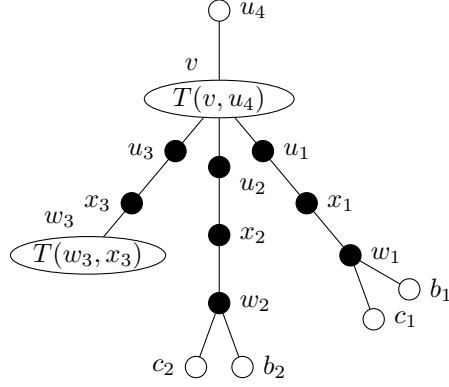


Figure 13: v is 2-linked to exactly one vertex of degree 4 in $T(v, u_4)$.

- . $\alpha(w_1) = \alpha(b_2)$.
 - $a = \alpha(w_1)$. Then apply: $x_3 \stackrel{T}{\leftarrow} a, v \leftarrow \alpha(w_2), u_3, x_1, u_1, u_2, x_2$.
 - $a \neq \alpha(w_1)$. Then apply: $x_3 \stackrel{T}{\leftarrow} a, v \leftarrow \alpha(w_1), u_3, x_2, u_2, u_1, x_1$.
- . $\alpha(b_1) = \alpha(b_2)$ (and $\alpha(w_1) \neq \alpha(w_2)$).
 - $a = \alpha(b_1)$. Then apply: $x_3 \stackrel{T}{\leftarrow} a, v \leftarrow \alpha(w_1), x_2, u_2, u_3, u_1, x_1$.
 - $a \neq \alpha(b_1)$. Then apply: $x_3 \stackrel{T}{\leftarrow} a, v \leftarrow \alpha(b_1), u_1, u_2, u_3, x_1, x_2$.
- 3. Vertex v is 2-linked in $T(v, u_4)$ to exactly two vertices w_2 and w_3 of degree 4. If $a = \alpha(w_1)$, then apply: $v \leftarrow \alpha(b_1)$, if $a \neq \alpha(w_1)$, then apply: $v \leftarrow \alpha(w_1)$. In both cases, we then color $x_3 \stackrel{T}{\leftarrow} a, x_2 \stackrel{T}{\leftarrow} a, u_2, u_3, u_1, x_1$.
- 4. Vertex v is 2-linked in $T(v, u_4)$ to three vertices w_1, w_2 and w_3 of degree 4. Apply: $x_1 \stackrel{T}{\leftarrow} a, x_2 \stackrel{T}{\leftarrow} a, x_3 \stackrel{T}{\leftarrow} a, v \leftarrow$ (the color of w_1), u_2, u_3, u_1 .

◇

Note that this claim can be extended to any coloring with more than 5 colors: we simply disregard the vertices that are colored by α outside a range C of 5 colors which contains a and $\alpha(w_i)$ (for $i = 1, 2, 3$). We apply the claim to the subgraph induced by the vertices colored within C : the extended coloring is compatible with the disregarded vertices.

Claim 5. G cannot contain (C_4) .

Proof. We deal with the worst-case situation, i.e. $d(u) = 3$: see Figure 14 for notations.

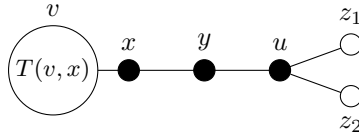


Figure 14: Worst case for claim (5).

Using the minimality of G , we have a coloring α of $G \setminus (T(v, x) \cup \{x, y\})$. We use claim (4) to extend it to the whole graph through: $x \stackrel{T}{\leftarrow} \alpha(z_1), y$.

◇

Claim 6. G cannot contain (C_5) .

Proof. Using the minimality of G , we get a coloring α of $G \setminus (T(v, x) \cup \{x, u, y_1, y_2\})$. Then we extend the coloring as follows:

- $\alpha(z_1) = \alpha(a_2)$. Then apply: $x \xleftarrow{T} \alpha(z_1), u, y_1, y_2$.
- $\alpha(z_1) \neq \alpha(a_2)$. Then apply: $x \xleftarrow{T} \alpha(z_1), u \leftarrow \alpha(a_2), y_1, y_2$.

This provides us with a coloring of the whole graph. ◇

□

A.2 Proof of Lemma 5

Proof. Assume by contradiction that there is a path $x - a - b - y$ such that both $T(x, a)$ and $T(y, b)$ exist. We consider without loss of generality that x and y are chosen such that $|T(y, b)|$ is minimum. By construction, $|T(y, b)| > 6$. Let b' be a neighbor of y . Then $T(y, b')$ exists (by definition, using the existence of $T(y, b)$). If there is no vertex $w \neq y$ of degree 4 that is 2-linked (with $w - c - b' - y$) to y , then the existence of $T(y, b')$ implies that the graph contains (C_4) , a contradiction. If such a w exists, $T(w, c)$ exists (by definition, using the existence of $T(y, b)$). Consequently, $y - b' - c - w$ is a path such that $T(y, b')$ and $T(w, c)$ exist. Additionally, $|T(w, c)| < |T(y, b)|$, a contradiction. □

A.3 Proof of Lemma 6

Proof. We attribute to each vertex a weight equal to its degree, and apply discharging rules R_1, R_2 and R_3 . We show that all the vertices have a weight of at least $\frac{7}{3}$ in the end.

There are no vertices of degree 0 or 1 in the graph, due to the fact that (C_1) is forbidden, so we study only the vertices of degree 2 or more.

Claim 7. All the vertices of degree 2 have a weight of at least $\frac{7}{3}$ after application of the rules.

Proof. Consider any vertex of degree 2 and let $x-s_1-\dots-s_p-y$, with $d(x), d(y) \geq 3$, be the unique branch that contains it. There is no Configuration (C_2) , so $p \leq 3$. According to the discharging rules, a vertex of degree 2 never gives away weight. We prove that it receives at least $\frac{1}{3}$. There are three cases depending on the value of p , each corresponding to Rule R_p :

- If $p = 1$, then Rule R_1 applies to $x-s_1-y$, and s_1 receives $\frac{1}{3}$.
- If $p = 2$, then Rule R_2 applies to $x-s_1-s_2-y$, and both s_1 and s_2 receive $\frac{1}{3}$.
- If $p = 3$, then since G does not contain Configuration (C_2) , $d(x), d(y) \geq 4$. Then, Rule R_3 applies to $x-s_1-s_2-s_3-y$, and s_1, s_2, s_3 receive $\frac{1}{3}$ each.

Consequently, each vertex of degree 2 starts with a weight of $\frac{6}{3}$, gives nothing away and receives at least $\frac{1}{3}$ during the discharging, which makes it end with a weight of at least $\frac{7}{3}$. ◇

Claim 8. All the vertices of degree 3 have a weight of at least $\frac{7}{3}$ after application of the rules.

Proof. We prove that a vertex v of degree 3 never gives away more than $\frac{2}{3}$. To each branch, it gives either $\frac{1}{3}$ [Rule $R_{2.1}$] or $\frac{1}{6}$ [Rules $R_{1.1}, R_{1.2}, R_{2.3}$] (or nothing). We prove that if v gives $\frac{1}{3}$ to two branches, then it gives nothing to the third. Assume, by contradiction, that v gives $\frac{1}{3}$ to two branches, and that the third one receives something from v . Since $R_{2.1}$ is the only rule that makes v give $\frac{1}{3}$ to a branch, it is applied twice. Then the third branch has to induce:

- A configuration for which $R_{1.1}$, $R_{2.1}$ or $R_{2.3}$ applies, i.e. a vertex of degree 2 followed by a vertex of degree at most 3. But then the graph contains (C_3) , a contradiction.
- A configuration for which $R_{1.2}$ applies. Then the graph contains (C_5) , a contradiction.

If v gives $\frac{1}{3}$ at most once, then v gives at most $\frac{2}{3}$ on the whole. So, in both cases, vertex v starts with a weight of $\frac{9}{3}$, and gives at most $\frac{2}{3}$ away, so it still has a weight of at least $\frac{7}{3}$ after application of the rules. \diamond

Claim 9. *All the vertices of degree 4 have a weight of at least $\frac{7}{3}$ after application of the rules.*

Proof.

We prove that a vertex v of degree 4 never gives more than $\frac{5}{3}$ away. To each branch, it gives either $\frac{1}{2}$ [Rules $R_{2.3}$, $R_{2.2}$, R_3], $\frac{1}{3}$ [Rules $R_{1.3}$, $R_{2.1}$] or $\frac{1}{6}$ [Rules $R_{1.1}$, $R_{1.2}$, $R_{2.3}$, $R_{2.2}$] (or nothing). We prove that if v gives $\frac{1}{2}$ thrice, then it gives at most $\frac{1}{6}$ to the fourth branch. Assume that v gives $\frac{1}{2}$ to three branches. The only case when v gives thrice $\frac{1}{2}$ is when for u_4 the other neighbor of v , $T(v, u_4)$ exists (we applied $R_{2.3}$, R_3 or $R_{2.2}$ on each of the three branch). Let us enumerate the cases for the branch starting from u_4 :

- Vertex u_4 is of degree at least 3. No rule applies, so v does not give anything to this branch.
- Vertex v is 1-linked to a vertex u of degree at least 3. If u is of degree ≥ 4 , then $R_{1.1}$ or $R_{1.3}$ applies. If u is of degree 3, then $R_{1.2}$ applies. In both cases, v does not give more than $\frac{1}{6}$.
- Vertex v is 2-linked to a vertex of degree at most 3. Then the graph contains (C_4) . Hence this case never occurs.
- Vertex v is 2-linked to a vertex u of degree at least 4. If u is of degree at least 5, then $R_{2.3}$ is applied. If u is of degree 4, then $R_{2.2}$ is applied (remember that due to Lemma 5, there is no ambiguity). In both cases, v does not give away more than $\frac{1}{6}$.

If v does not give $\frac{1}{2}$ more than twice, then v gives at most $\frac{5}{3}$ on the whole. So, in both cases, v starts with a weight of $\frac{12}{3}$, and gives at most $\frac{5}{3}$ away, so it still has a weight of at least $\frac{7}{3}$ after application of the rules. \diamond

Claim 10. *All the vertices of degree ≥ 5 have a weight of at least $\frac{7}{3}$ after application of the rules.*

Proof. Each vertex gives at most $\frac{1}{2}$ to each branch. Hence, a vertex v gives at most $d(v) \times \frac{1}{2}$ on the whole. And for $d(v) \geq 5$, we have $d(v) - \frac{1}{2} \times d(v) \geq \frac{7}{3}$. \diamond

Hence every vertex of G has a weight of at least $\frac{7}{3}$ after application of the discharging rules. Consequently, $\frac{\sum_{v \in V} d(v)}{|V|} \geq \frac{7}{3}$, which implies $\text{mad}(G) \geq \frac{7}{3}$. \square

B Proof of Theorem 5

B.1 Proof of Lemma 7

Before proving the lemma, we need the following useful claim about the structure of a $T(v, u_1)$.

Claim 11. *Let H be a graph that contains a $T(v, u_1)$. For any 2-distance $(k+1)$ -list-coloring α of $H \setminus (T(v, u_1) \cup N(u_1))$ such that two vertices that have a common neighbor in H have distinct colors, the coloring α can be extended to $(H \setminus N(u_1)) \cup \{v\}$.*

In the description of a coloring procedure, we denote " $u_1 \xleftarrow{T} c$ " as a shortcut for "We color u_1 in c and apply Claim 11 to color $T(v, u_1)$ ", and we denote also " x " as a shortcut for "We color arbitrarily x with one of the colors available, and apply Claim 11 if necessary".

Proof. We pick $b \in L(v)$, $b = \alpha(u_1)$ if it belongs to $L(v)$.

For every w_i such that $T(b_i, w_i)$ exists and at most $(k - 2)$ vertices a_j have already been colored, for $c \in L(b_i)$ such that $c = b$ or $c \notin L(v)$, $b_i \stackrel{T}{\leftarrow} c, a_i$.

If $T(b_i, w_i)$ exists for every w_i , let us call b_2 the one which is not colored yet. We call A the set of colors used on the vertices a_j that are already colored. Note that $|A| = k - 1$. If $\exists c, c \in A, c \in L(b_2)$, then $b_2 \stackrel{T}{\leftarrow} c, a_2, v$. If no such c exists, we are in one of the following situations:

- $\exists c, c \in A, c \notin L(a_2). b_2, v, a_2.$
- $A \subseteq L(a_2).$
 - $\exists c, c \in A, c \notin L(v). b_2 \stackrel{T}{\leftarrow} d \notin L(a_2), a_2, v.$
 - $A \subseteq L(v)$. Since $A \cap L(b_2) = \emptyset, |A| = k - 1$ and $|L(v)| = |L(a_2)| = |L(b_2)| = k + 1, |L(v) \cap L(b_2)| \leq 2, |L(a_2) \cap L(b_2)| \leq 2$. But $k > 3$, hence there exists $c \in L(b_2)$ such that $c \notin L(v), c \notin L(a_2)$. Then $b_2 \stackrel{T}{\leftarrow} c, a_2, v.$

Then, for every i such that $T(b_i, w_i)$ exists, we color a_i (at most $d(v) - |\{v\}| + |\{w_i\}| \leq k$ constraints). We color v . Then we color the remaining vertices b_i (at most $d(w_i) + 2 \leq k$ constraints). \diamond

Proof.

Claim 12. G cannot contain (C_1)

Proof. Using the minimality of G , we color $G \setminus \{u\}$. Since $\Delta(G) \leq k$, and $d(u) \leq 1$, vertex u has at most k constraints. There are $k + 1$ colors, so the coloring of $G \setminus \{u\}$ can be extended to G . \diamond

Claim 13. G cannot contain (C_2)

Proof. Using the minimality of G , we color $G \setminus \{v, u_2\}$. Vertex u_2 has at most $|\{w_2, u_1\}| + d(w_2) - 1 = d(w_2) + 1 \leq k$ constraints. Hence we can color u_2 . Then v has at most $4 \leq k$ constraints, so we can extend the coloring of $G \setminus \{v, u_1\}$ to G . \diamond

Claim 14. G cannot contain (C_3)

Proof. Using the minimality of G , we color $G \setminus \{u_1, u_2\}$. Vertex u_1 has at most $|\{w_2\}| + d(w_1) \leq 1 + (k - 1) \leq k$ constraints. Hence we can color u_1 . Then u_2 has at most $|\{w_1, u_1\}| + d(w_2) \leq 2 + (k - 2) \leq k$ constraints, so we can extend the coloring of $G \setminus \{u_1, u_2\}$ to G . \diamond

Claim 15. G cannot contain (C_4)

Proof. Using the minimality of G , we color $G \setminus (T(w_1, u_1) \cup \{u_1, u_2\})$. $u_1, u_2.$ \diamond

\square

Lemma 10. For $k > 5$, if G is a minimal graph such that $\Delta(G) \leq k$ and G admits no 2-distance $(k + 1)$ -list-coloring, then G cannot contain Configuration (C_5) .

Proof. Using the minimality of G , we color $G \setminus \{u_1\}$ ($\setminus T(w_1, u_1)$ if it exists). We discolor $v, u_1, v.$ \square

Lemma 11. For $k > 8$, if G is a minimal graph such that $\Delta(G) \leq k$ and G admits no 2-distance $(k + 1)$ -list-coloring, then G cannot contain Configuration (C_6) .

Proof. Using the minimality of G , we color $G \setminus \{u_2\}$ ($\setminus T(w_2, u_2)$ if it exists). We discolor $v, u_1, v.$ \square

B.2 proof of Lemma 8

Proof. Assume by contradiction that there is a path $x - a - b - y$ such that both $T(x, a)$ and $T(y, b)$ exist. We consider without loss of generality that x and y are chosen such that $|T(y, b)|$ is minimum. By construction, $|T(y, b)| > 6$. Let b' be a neighbor of y . Then $T(y, b')$ exists (by definition, using the existence of $T(y, b)$). If there is no vertex $w \neq y$ of degree k that is 2-linked (with $w - c - b' - y$) to y , then the existence of $T(y, b')$ implies that the graph contains (C_4) , a contradiction. If such a w exists, $T(w, c)$ exists (by definition, using the existence of $T(y, b)$). Consequently, $y - b' - c - w$ is a path such that $T(y, b')$ and $T(w, c)$ exist. Additionally, $|T(w, c)| < |T(y, b)|$, a contradiction. \square

B.3 Proof of Lemma 9

Proof. We attribute to each vertex a weight equal to its degree, and apply discharging rules R_1, R_2, R_3 and R_4 . We show that all the vertices have a weight of at least $\frac{8}{3} - \epsilon$ in the end.

We first note that since $k > 4$, $k - 1 \geq M_2$. Indeed, $M_2 = \frac{8 - \frac{4}{k}}{1 + \frac{4}{k}} \leq k - 1$ when $k \geq 5$.

We also note that when $k \leq 8$, $M_2 \leq 5$, and that for any k , $M_1 \leq 4$ and $M_2 \leq 8$.

- There are no vertices of degree 0 or 1.
- Let s be a maximal path of vertices of degree 2 (maximal in the sense that it does not admit a vertex of degree 2 as a neighbor). According to the discharging rules, a vertex of degree 2 never gives away weight. We prove that it receives at least $\frac{2}{3} - \epsilon$. There are three cases depending on the size of s (s can't be of size greater than 3 due to Configuration (C_2)):
 - $|s| = 1$. Let a be the only vertex in s .
 - * a has a neighbor x of degree at least M_2 such that $T(x, a)$ does not exist: then it receives $\frac{2}{3} - \epsilon$ from it, according to Rule R_3 or $R_4(i)$.
 - * a has two neighbors x_1 and x_2 of degree less than M_2 or such that $T(x_i, a)$ exists: then it receives $\frac{1}{3} - \frac{\epsilon}{2}$ from each, according to Rule R_1, R_2 or $R_4(i)'$.
 - $|s| = 2$. Let a and b be the vertices of s , and x (resp. y) the other neighbor of a (resp. b).
 - * $d(y) < M_2$ (so $d(y) < k - 1$) or $T(y, b)$ exists. Then, due to Configurations (C_3) , (C_4) and Lemma 8, $d(x) = k$ and $T(x, a)$ does not exist. Then b receives $\frac{1}{3} - \frac{\epsilon}{2}$ from y (Rule R_1 or R_2), and from x (Rule $R_4(ii)$), and a receives $\frac{2}{3} - \epsilon$ from x (Rule $R_4(i)$).
 - * $d(y), d(x) \geq M_2$ and neither $T(x, a)$ nor $T(y, b)$ exists. Then, according to Rule R_3 or $R_4(i)$, a (resp. b) receives $\frac{2}{3} - \epsilon$ from x (resp. y).
 - $|s| = 3$. Due to Configuration (C_4) , for $a_1 - b - a_2$ the vertices of s and x_i the other neighbor of a_i , $d(x_1) = d(x_2) = k$ and no $T(x_i, a_i)$ exists. Then Rules $R_4(i)$ and $R_4(ii)$ apply: a_i receives $\frac{2}{3} - \epsilon$ from x_i and b receives $\frac{1}{3} - \frac{\epsilon}{2}$ from both x_1 and x_2 .
- Let x be a vertex with $3 \leq d(x) < M_1$. If such an x exists, then $M_1 > 3$, so $k > 5$. And since $M_1 \leq 4$, $d(x) = 3$. We prove that x gives at most $\frac{1}{3} + \epsilon$ away.
 - If $6 \leq k \leq 8$, $\epsilon \geq \frac{1}{6}$ and G cannot contain Configuration (C_5) . Only Rule R_1 can apply to x , so x gives at most $\frac{1}{3} - \frac{\epsilon}{2}$ to each of its neighbors. Since $\epsilon \geq \frac{1}{6}$, $\frac{1}{3} + \epsilon \geq 2 * (\frac{1}{3} - \frac{\epsilon}{2})$. We prove that if it gives $\frac{1}{3} - \frac{\epsilon}{2}$ to two of its neighbors, then it gives nothing to the third one. Assume, for contradiction, that x gives $\frac{1}{3} - \frac{\epsilon}{2}$ to each of its three neighbors (meaning that x is 1-linked (by a_1, a_2 or a_3) to three vertices y_1, y_2, y_3 of degree at most $M_2 - 1$ or such that $T(y_i, a_i)$ exists). Since $k \leq 8$, $M_2 \leq 5$. Consequently, this is a case of Configuration (C_5) , a contradiction.
 - If $k \geq 9$, G cannot contain Configuration (C_6) . Only Rule R_1 can apply to x , so x gives at most $\frac{1}{3} - \frac{\epsilon}{2}$ to each of its neighbors. Here, $\frac{1}{3} + \epsilon \geq \frac{1}{3} - \frac{\epsilon}{2}$. Assume that x gives $\frac{1}{3} - \frac{\epsilon}{2}$ to two neighbors $a_1 a_2$ whose other neighbors are y_1 and y_2 . Let a_3 be the third neighbor of x .

- * $d(a_3) \leq 3$. Then, since $M_2 \leq 8$, this is a case of Configuration (C_6) , a contradiction.
- * $d(a_3) \geq 4$. Then, due to Rule $R_2, R_3, R_4(i)$ or $R_4(i)'$, a_3 gives at least $\frac{1}{3} - \frac{\epsilon}{2}$ to x so that on the whole, x loses a weight of at most $\frac{1}{3} - \frac{\epsilon}{2} \leq \frac{1}{3} + \epsilon$.
- Let x be a vertex with $M_1 \leq d(x) < M_2$. By definition of M_1 and R_2 , x gives never more than $d(x) - \frac{8}{3} + \epsilon$ away.
- Let x be a vertex with $M_2 \leq d(x) < k$. By definition of M_2 and R_3 , x gives never more than $d(x) - \frac{8}{3} + \epsilon$ away.
- Let x be a vertex with $d(x) = k$. By definition of ϵ , if x gives at most $(k-1) \times (1 - \frac{3\epsilon}{2}) + \frac{1}{3} - \frac{\epsilon}{2} = (k-2) \times (1 - \frac{3\epsilon}{2}) + 2 \times (\frac{2}{3} - \epsilon)$ away, then it gives at most $d(x) - \frac{8}{3} + \epsilon$ away. Assume that x gives $(k-1)$ times $(1 - \frac{3\epsilon}{2})$ (meaning that both $R_4(i)$ and $R_4(ii)$ are applied $(k-1)$ times). Then, for u_1 the remaining neighbor of x , $T(x, u_1)$ exists and x gives on the whole $(k-1) \times (1 - \frac{3\epsilon}{2}) + \frac{1}{3} - \frac{\epsilon}{2}$.

Consequently, after application of the discharging rules, every vertex v of G has a weight of at least $\frac{8}{3} - \epsilon$, meaning that $\sum_{v \in G} d(v) \geq \sum_{v \in G} (\frac{8}{3} - \epsilon)$. Therefore, $\text{mad}(G) \geq \frac{8}{3} - \epsilon$ □