Ontologies and Description Logics

Parcours IA - Représentation des connaissances

Meghyn Bienvenu, CNRS researcher at LaBRI
Reasoning Techniques for Expressive DLs
Tableau method

Tableau method: most popular approach to reasoning in expressive DLs
▶ implemented in state-of-the-art DL reasoners

Tableau algorithms are used to **decide satisfiability**.
▶ can also be used for other reasoning tasks (e.g. instance checking) that can be reduced to satisfiability
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Idea: to determine whether a given (concept or KB) $\Psi$ is satisfiable, try to construct a (representation of a) model of $\Psi$
▶ if we succeed, then we have shown that $\Psi$ is satisfiable
▶ if we fail despite having considered all possibilities, then we have proven that $\Psi$ is unsatisfiable
**ALC-concepts**

Recall that ALC-concepts are built using the following constructors:

\[ \top \quad \bot \quad \neg \quad \sqcup \quad \sqcap \quad \forall R.C \quad \exists R.C \]
**ALC-concepts**

Recall that *ALC*-concepts are built using the following constructors:

\[ \bot \quad \top \quad \neg \quad \sqcup \quad \sqcap \quad \forall R.C \quad \exists R.C \]

We say that an *ALC*-concept $C$ is in **negation normal form (NNF)** if the symbol $\neg$ only appears directly in front of atomic concepts.

- **in NNF:** $A \sqcap \neg B$, $\exists R.\neg A$, $\neg A \sqcup \neg B$

- **not in NNF:** $\neg(A \sqcap B)$, $\exists R.\neg(\forall S.B)$, $A \sqcup \neg\forall R.B$, $\neg\top$
ALC-concepts

Recall that ALC-concepts are built using the following constructors:

⊤ ⊥ ¬ ⊔ ⊓ ∀R.C ∃R.C

We say that an ALC-concept C is in negation normal form (NNF) if the symbol ∼ only appears directly in front of atomic concepts.

▶ in NNF: A ⊓ ¬B, ∃R.¬A, ¬A ⊔ ¬B

▶ not in NNF: ¬(A ⊓ B), ∃R.¬(∀S.B), A ⊔ ¬∀R.B, ¬⊤

Fact. Every ALC-concept C can be transformed into an equivalent concept in NNF in linear time by applying the following rewriting rules:

¬⊤ → ⊥ ¬(C ⊓ D) → ¬C ⊔ ¬D ¬(∀R.C) → ∃R.¬C
¬⊥ → ⊤ ¬(C ⊔ D) → ¬C ⊓ ¬D ¬(∃R.C) → ∀R.¬C

Note: say C and D are equivalent if the empty TBox entails C ≡ D.
Algorithm for computing NNF

**Algorithm NNF**

**Input:** \(\mathcal{ALC}\)-concept \(C\)

- If \(C = \top\) or \(C = \bot\), then output \(C\)
- If \(C = A\) or \(C = \neg A\) (with \(A\) atomic concept), then output \(C\)
- If \(C = D_1 \sqcap D_2\), then output \(\text{NNF}(D_1) \sqcap \text{NNF}(D_2)\)
- If \(C = D_1 \sqcup D_2\), then output \(\text{NNF}(D_1) \sqcup \text{NNF}(D_2)\)
- If \(C = \exists R . D\), then output \(\exists R . \text{NNF}(D)\)
- If \(C = \forall R . D\), then output \(\forall R . \text{NNF}(D)\)
- If \(C = \neg \top\), return \(\bot\); if \(C = \neg \bot\), then output \(\top\)
- If \(C = \neg(D_1 \sqcap D_2)\), then output \(\text{NNF}(\neg D_1) \sqcup \text{NNF}(\neg D_2)\)
- If \(C = \neg(D_1 \sqcup D_2)\), then output \(\text{NNF}(\neg D_1) \sqcap \text{NNF}(\neg D_2)\)
- If \(C = \neg \exists R . D\), then output \(\forall R . \text{NNF}(\neg D)\)
- If \(C = \neg \forall R . D\), then output \(\exists R . \text{NNF}(\neg D)\)
- If \(C = \neg(\neg D)\), then output \(\text{NNF}(\neg D)\)

(we use \(\text{NNF}(E)\) to denote output of NNF on input \(E\))
Satisfiability of $\mathcal{ALC}$-concepts via tableau

We start by giving a tableau algorithm for deciding satisfiability of $\mathcal{ALC}$-concepts in NNF w.r.t. the empty TBox.

**Procedure for testing satisfiability of $C_0$:**

- We work with a set $S$ of ABoxes
- Initially, $S$ contains a single ABox $\{C_0(a_0)\}$
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Procedure for testing satisfiability of \( C_0 \):

- We work with a set \( S \) of ABoxes
- Initially, \( S \) contains a single ABox \( \{C_0(a_0)\} \)
- At each stage, we apply a rule to some \( A \in S \)
  (note: rules are detailed on next slide)
We start by giving a tableau algorithm for deciding satisfiability of $\mathcal{ALC}$-concepts in NNF w.r.t. the empty TBox.

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- Stop applying rules when either:
  - every $A \in S$ contains a clash, i.e. an assertion $\bot(b)$ or a pair of assertions $\{B(b), \neg B(b)\}$
  - some $A \in S$ is clash-free and complete: no rule can be applied to $A$
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  - some $A \in S$ is clash-free and complete: no rule can be applied to $A$
- Return “yes” if some $A \in S$ is clash-free, else “no”.


Tableau rules for $\mathcal{ALC}$

$\sqcap$-rule: if $(C_1 \sqcap C_2)(a) \in \mathcal{A}$ and $\{C_1(a), C_2(a)\} \not\subseteq \mathcal{A}$
then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a), C_2(a)\}$
Tableau rules for $\mathcal{ALC}$

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$\sqcup$-rule: if $(C_1 \sqcup C_2)(a) \in \mathcal{A}$ and $\{C_1(a), C_2(a)\} \cap \mathcal{A} = \emptyset$
then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a)\}$ and $\mathcal{A} \cup \{C_2(a)\}$
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$\forall$-rule: if $\{\forall R.C(a), R(a, b)\} \in \mathcal{A}$ and $C(b) \not\in \mathcal{A}$
then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C(b)\}$
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$\forall$-rule: if $\{\forall R. C(a), R(a, b)\} \in \mathcal{A}$ and $C(b) \not\in \mathcal{A}$
then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C(b)\}$

$\exists$-rule: if $\{\exists R. C(a)\} \in \mathcal{A}$ and there is no $b$ with $\{R(a, b), C(b)\} \subseteq \mathcal{A}$
then pick a new individual name $d$ and
replace $\mathcal{A}$ with $\mathcal{A} \cup \{R(a, d), C(d)\}$
Tableau example: only $\square$ and $\Box$

Let’s use the tableau procedure to test satisfiability of

$$C_0 = (A \boxplus B) \square ((\neg B \boxplus D) \square \neg A)$$
Tableau example: only $\square$ and $\blacksquare$

Let’s use the tableau procedure to test satisfiability of

$$C_0 = (A \square B) \cap ((\neg B \square D) \cap \neg A)$$

Start with $S = \{ A_0 \}$ where $A_0 = \{ ( (A \square B) \cap ((\neg B \square D) \cap \neg A) ) (a_0) \}$. 
Let’s use the tableau procedure to test satisfiability of
\[ C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ ((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)) (a_0) \} \).

Apply \( \sqcap \)-rule to \( A_0 \):
get \( S = \{ A'_0 \} \) where \( A'_0 = A_0 \cup \{ (A \sqcup B)(a_0), ((\neg B \sqcup D) \sqcap \neg A)(a_0) \} \).
Tableau example: only \(\square\) and \(\Box\)

Let’s use the tableau procedure to test satisfiability of

\[C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\]

Start with \(S = \{ \mathcal{A}_0 \}\) where \(\mathcal{A}_0 = \{( (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) ) (a_0) \}\).

**Apply \(\square\)-rule to \(\mathcal{A}_0\):**
get \(S = \{ \mathcal{A}'_0 \}\) where \(\mathcal{A}'_0 = \mathcal{A}_0 \cup \{(A \sqcup B)(a_0), ((\neg B \sqcup D) \sqcap \neg A)(a_0)\}\).
Tableau example: only $\Box$ and $\bigvee$

Let’s use the tableau procedure to test satisfiability of

$$C_0 = (A \Box B) \bigcap ((\neg B \bigvee D) \bigcap \neg A)$$

Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ ((A \bigvee B) \bigcap ((\neg B \bigvee D) \bigcap \neg A))(a_0) \}$. 

**Apply $\bigcap$-rule to $\mathcal{A}_0$:**
get $S = \{ \mathcal{A}_0' \}$ where $\mathcal{A}_0' = \mathcal{A}_0 \cup \{(A \bigvee B)(a_0), ((\neg B \bigvee D) \bigcap \neg A)(a_0)\}$.

**Apply $\bigvee$-rule to $\mathcal{A}_0'$:**
get $S = \{ \mathcal{A}_1, \mathcal{A}_2 \}$ where $\mathcal{A}_1 = \mathcal{A}_0' \cup \{A(a_0)\}$ and $\mathcal{A}_2 = \mathcal{A}_0' \cup \{B(a_0)\}$. 
Let’s use the tableau procedure to test satisfiability of

\[ C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ ((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A))(a_0) \} \).

**Apply \( \sqcap \)-rule to \( A_0 \):**
get \( S = \{ A'_0 \} \) where \( A'_0 = A_0 \cup \{ (A \sqcup B)(a_0), ((\neg B \sqcup D) \sqcap \neg A)(a_0) \} \).

**Apply \( \sqcup \)-rule to \( A'_0 \):**
get \( S = \{ A_1, A_2 \} \) where \( A_1 = A'_0 \cup \{ A(a_0) \} \) and \( A_2 = A'_0 \cup \{ B(a_0) \} \).
Let's use the tableau procedure to test satisfiability of

\[ C_0 = (A \cup B) \cap ((\neg B \cup D) \cap \neg A) \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ ((A \cup B) \cap ((\neg B \cup D) \cap \neg A))(a_0) \} \).

**Apply \( \cap \)-rule to \( A_0 \):**
get \( S = \{ A'_0 \} \) where \( A'_0 = A_0 \cup \{ (A \cup B)(a_0), ((\neg B \cup D) \cap \neg A)(a_0) \} \).

**Apply \( \cup \)-rule to \( A'_0 \):**
get \( S = \{ A_1, A_2 \} \) where \( A_1 = A'_0 \cup \{ A(a_0) \} \) and \( A_2 = A'_0 \cup \{ B(a_0) \} \).

**Apply \( \cap \)-rule to \( A_1 \):**
get \( S = \{ A'_1, A_2 \} \) where \( A'_1 = A_1 \cup \{ (\neg B \cup D)(a_0), \neg A(a_0) \} \).
Tableau example: only $\Box$ and $\square$

Let’s use the tableau procedure to test satisfiability of

$$C_0 = (A \sqcup B) \cap ((\neg B \sqcup D) \cap \neg A)$$

Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ ((A \sqcup B) \cap ((\neg B \sqcup D) \cap \neg A))(a_0) \}$.  

Apply $\Box$-rule to $\mathcal{A}_0$:
get $S = \{ \mathcal{A}_0' \}$ where $\mathcal{A}_0' = \mathcal{A}_0 \cup \{(A \sqcup B)(a_0), ((\neg B \sqcup D) \cap \neg A)(a_0) \}$.  

Apply $\sqcup$-rule to $\mathcal{A}_0'$:
get $S = \{ \mathcal{A}_1, \mathcal{A}_2 \}$ where $\mathcal{A}_1 = \mathcal{A}_0' \cup \{A(a_0)\}$ and $\mathcal{A}_2 = \mathcal{A}_0' \cup \{B(a_0)\}$.  

Apply $\Box$-rule to $\mathcal{A}_1$:
get $S = \{ \mathcal{A}_1', \mathcal{A}_2 \}$ where $\mathcal{A}_1' = \mathcal{A}_1 \cup \{\neg B \sqcup D)(a_0), \neg A(a_0)\}$  

$\mathcal{A}_1'$ contains clash $\{A(a_0), \neg A(a_0)\}$!
Let’s use the tableau procedure to test satisfiability of

\[ C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ ((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A))(a_0) \} \).

**Apply \( \sqcap \)-rule to \( A_0 \):**
get \( S = \{ A_0' \} \) where \( A_0' = A_0 \cup \{ (A \sqcup B)(a_0), ((\neg B \sqcup D) \sqcap \neg A)(a_0) \} \).

**Apply \( \sqcup \)-rule to \( A_0' \):**
get \( S = \{ A_1, A_2 \} \) where \( A_1 = A_0' \cup \{ A(a_0) \} \) and \( A_2 = A_0' \cup \{ B(a_0) \} \).

**Apply \( \sqcap \)-rule to \( A_1 \):**
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**Apply \( \sqcap \)-rule to \( A_0 \):**
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**Apply \( \sqcup \)-rule to \( A'_0 \):**
get \( S = \{ A_1, A_2 \} \) where \( A_1 = A'_0 \cup \{A(a_0)\} \) and \( A_2 = A'_0 \cup \{B(a_0)\} \).

**Apply \( \sqcap \)-rule to \( A_1 \):**
get \( S = \{ A'_1, A_2 \} \) where \( A'_1 = A_1 \cup \{\neg B \sqcup D)(a_0), \neg A(a_0)\} \)

**Apply \( \sqcap \)-rule to \( A_2 \):**
get \( S = \{ A'_1, A'_2 \} \) where \( A'_2 = A_2 \cup \{\neg B \sqcup D)(a_0), \neg A(a_0)\} \)
Tableau example: only □ and ◯

Let’s use the tableau procedure to test satisfiability of

\[ C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ ((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A))(a_0) \} \).

**Apply □-rule to \( A_0 \):**
get \( S = \{ A'_0 \} \) where \( A'_0 = A_0 \cup \{ (A \sqcup B)(a_0), ((\neg B \sqcup D) \sqcap \neg A)(a_0) \} \).

**Apply □-rule to \( A'_0 \):**
get \( S = \{ A_1, A_2 \} \) where \( A_1 = A'_0 \cup \{ A(a_0) \} \) and \( A_2 = A'_0 \cup \{ B(a_0) \} \).

**Apply □-rule to \( A_1 \):**
get \( S = \{ A'_1, A_2 \} \) where \( A'_1 = A_1 \cup \{ (\neg B \sqcup D)(a_0), \neg A(a_0) \} \)

**Apply □-rule to \( A_2 \):**
get \( S = \{ A'_1, A'_2 \} \) where \( A'_2 = A_2 \cup \{ (\neg B \sqcup D)(a_0), \neg A(a_0) \} \)
Let's use the tableau procedure to test satisfiability of
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Apply \( \sqcup \)-rule to \( A'_0 \):
get \( S = \{ A_1, A_2 \} \) where \( A_1 = A'_0 \cup \{ A(a_0) \} \) and \( A_2 = A'_0 \cup \{ B(a_0) \} \).

Apply \( \sqcap \)-rule to \( A_1 \):
get \( S = \{ A'_1, A_2 \} \) where \( A'_1 = A_1 \cup \{ (\neg B \sqcup D)(a_0), \neg A(a_0) \} \}

Apply \( \sqcap \)-rule to \( A_2 \):
get \( S = \{ A'_2 \} \) where \( A'_2 = A_2 \cup \{ (\neg B \sqcup D)(a_0), \neg A(a_0) \} \}

Apply \( \sqcup \)-rule to \( A'_2 \):
get \( S = \{ A'_1, A_3, A_4 \} \) where \( A_3 = A'_2 \cup \{ \neg B(a_0) \} \), \( A_4 = A'_2 \cup \{ D(a_0) \} \)
Let’s use the tableau procedure to test satisfiability of

\[ C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ ((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A))(a_0) \} \).

**Apply \( \sqcap \)-rule to \( A_0 \):**
get \( S = \{ A'_0 \} \) where \( A'_0 = A_0 \cup \{ (A \sqcup B)(a_0), ((\neg B \sqcup D) \sqcap \neg A)(a_0) \} \).

**Apply \( \sqcup \)-rule to \( A'_0 \):**
get \( S = \{ A_1, A_2 \} \) where \( A_1 = A'_0 \cup \{ A(a_0) \} \) and \( A_2 = A'_0 \cup \{ B(a_0) \} \).

**Apply \( \sqcap \)-rule to \( A_1 \):**
get \( S = \{ A'_1, A_2 \} \) where \( A'_1 = A_1 \cup \{ (\neg B \sqcup D)(a_0), \neg A(a_0) \} \)

**Apply \( \sqcap \)-rule to \( A_2 \):**
get \( S = \{ A'_1, A'_2 \} \) where \( A'_2 = A_2 \cup \{ (\neg B \sqcup D)(a_0), \neg A(a_0) \} \)

**Apply \( \sqcup \)-rule to \( A'_2 \):**
get \( S = \{ A'_1, A_3, A_4 \} \) where \( A_3 = A'_2 \cup \{ \neg B(a_0) \}, A_4 = A'_2 \cup \{ D(a_0) \} \)

\( A_3 \) contains clash \( \{ B(a_0), \neg B(a_0) \} \)!
Tableau example: only □ and ▴

Let’s use the tableau procedure to test satisfiability of

\[ C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) \]

Start with \( S = \{ \mathcal{A}_0 \} \) where \( \mathcal{A}_0 = \{ ((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A))(a_0) \} \).

**Apply □-rule to \( \mathcal{A}_0 \):**
get \( S = \{ \mathcal{A}_0' \} \) where \( \mathcal{A}_0' = \mathcal{A}_0 \cup \{(A \sqcup B)(a_0), ((\neg B \sqcup D) \sqcap \neg A)(a_0)\} \).

**Apply ▴-rule to \( \mathcal{A}_0' \):**
get \( S = \{ \mathcal{A}_1, \mathcal{A}_2 \} \) where \( \mathcal{A}_1 = \mathcal{A}_0' \cup \{A(a_0)\} \) and \( \mathcal{A}_2 = \mathcal{A}_0' \cup \{B(a_0)\} \).

**Apply □-rule to \( \mathcal{A}_1 \):**
get \( S = \{ \mathcal{A}_1', \mathcal{A}_2 \} \) where \( \mathcal{A}_1' = \mathcal{A}_1 \cup \{\neg B \sqcup D)(a_0), \neg A(a_0)\} \)

**Apply □-rule to \( \mathcal{A}_2 \):**
get \( S = \{ \mathcal{A}_1', \mathcal{A}_2' \} \) where \( \mathcal{A}_2' = \mathcal{A}_2 \cup \{\neg B \sqcup D)(a_0), \neg A(a_0)\} \)

**Apply ▴-rule to \( \mathcal{A}_2' \):**
get \( S = \{ \mathcal{A}_1', \mathcal{A}_3, \mathcal{A}_4 \} \) where \( \mathcal{A}_3 = \mathcal{A}_2' \cup \{\neg B(a_0)\}, \mathcal{A}_4 = \mathcal{A}_2' \cup \{D(a_0)\} \)

\( \mathcal{A}_4 \) is complete, so we can stop.
Tableau example: only $\Box$ and $\sqcap$

Let’s use the tableau procedure to test satisfiability of

$$C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)$$

Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ ((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A))(a_0) \}$.

**Apply $\Box$-rule to $\mathcal{A}_0$:**
get $S = \{ \mathcal{A}'_0 \}$ where $\mathcal{A}'_0 = \mathcal{A}_0 \cup \{(A \sqcup B)(a_0), ((\neg B \sqcup D) \sqcap \neg A)(a_0)\}$.

**Apply $\sqcap$-rule to $\mathcal{A}'_0$:**
get $S = \{ \mathcal{A}_1, \mathcal{A}_2 \}$ where $\mathcal{A}_1 = \mathcal{A}'_0 \cup \{A(a_0)\}$ and $\mathcal{A}_2 = \mathcal{A}'_0 \cup \{B(a_0)\}$.

**Apply $\Box$-rule to $\mathcal{A}_1$:**
get $S = \{ \mathcal{A}'_1, \mathcal{A}_2 \}$ where $\mathcal{A}'_1 = \mathcal{A}_1 \cup \{\neg B \sqcup D)(a_0), \neg A(a_0)\}$

**Apply $\Box$-rule to $\mathcal{A}_2$:**
get $S = \{ \mathcal{A}'_1, \mathcal{A}'_2 \}$ where $\mathcal{A}'_2 = \mathcal{A}_2 \cup \{\neg B \sqcup D)(a_0), \neg A(a_0)\}$

**Apply $\sqcap$-rule to $\mathcal{A}'_2$:**
get $S = \{ \mathcal{A}'_1, \mathcal{A}_3, \mathcal{A}_4 \}$ where $\mathcal{A}_3 = \mathcal{A}'_2 \cup \{\neg B(a_0)\}$, $\mathcal{A}_4 = \mathcal{A}'_2 \cup \{D(a_0)\}$

$\mathcal{A}_4$ contains no clash $\Rightarrow C_0$ is satisfiable
Previous example in a picture

\[(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)(a_0)\]
\[(A \sqcap B) \sqcap ((\neg B \sqcap D) \sqcap \neg A) (a_0)\]
\[\quad \quad ((\neg B \sqcap D) \sqcap \neg A) (a_0)\]
\[\quad \quad (A \sqcap B) (a_0)\] \hspace{1cm} \square\text{-rule}
\[(A \sqcup B) \cap ((\neg B \sqcup D) \cap \neg A) (a_0)\]
\[\cap ((\neg B \sqcup D) \cap \neg A) (a_0)\]
\[((\neg B \sqcup D) \cap \neg A) (a_0)\]
\[(A \sqcup B) (a_0)\]

\[A (a_0) \quad \square \quad B (a_0) \quad \square\text{-rule}\]
Previous example in a picture

\[(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) (a_0)\]
\[((\neg B \sqcup D) \sqcap \neg A) (a_0)\]
\[(A \sqcup B) (a_0)\]

\[A (a_0)\]
\[\sqcap\text{-rule}\]
\[\neg B \sqcup D) (a_0)\]
\[\neg A (a_0)\]

\[B (a_0)\]
\[(A \sqcup B) \cap ((\neg B \sqcup D) \cap \neg A)(a_0)\]
\[((\neg B \sqcup D) \cap \neg A)(a_0)\]
\[(A \sqcup B)(a_0)\]

\[
\begin{align*}
A(a_0) & \quad & B(a_0) \\
(\neg B \sqcup D)(a_0) & \quad & (\neg B \sqcup D)(a_0) \\
\neg A(a_0) & \quad & \neg A(a_0)
\end{align*}
\]

\[\sqcap\text{-rule}\]
Previous example in a picture

\[(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) (a_0)\]
\[((\neg B \sqcup D) \sqcap \neg A) (a_0)\]
\[(A \sqcup B) (a_0)\]

\[A (a_0)\]
\[\neg B (a_0)\]
\[\neg A (a_0)\]

\[B (a_0)\]
\[\neg B (a_0)\]
\[\neg A (a_0)\]

\[\neg B (a_0)\]
\[D (a_0)\]

\[\sqcap\text{-rule}\]
(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) (a_0)
((\neg B \sqcup D) \sqcap \neg A) (a_0)
(A \sqcup B) (a_0)

A (a_0)
(\neg B \sqcup D) (a_0)
\neg A (a_0)

B (a_0)
(\neg B \sqcup D) (a_0)
\neg A (a_0)
\neg B (a_0)
D (a_0)
Previous example in a picture

\[
(A \sqcup B) \cap (\neg B \sqcup D) \cap \neg A \quad (a_0)
\]

\[
((\neg B \sqcup D) \cap \neg A) \quad (a_0)
\]

\[
(A \sqcup B) \quad (a_0)
\]

\[
A \quad (a_0)
\]

\[
\neg B \quad (a_0)
\]

\[
\neg A \quad (a_0)
\]

\[
\neg B \quad (a_0)
\]

\[
\neg A \quad (a_0)
\]

\[
\neg B \quad (a_0)
\]

\[
D \quad (a_0)
\]

* ✔

* ✘
**Previous example in a picture**

\[ A_1' \]

\[
(\overline{A} \lor B) \land ((\overline{B} \lor D) \land \overline{A}) (a_0) \\
((\overline{B} \lor D) \land \overline{A}) (a_0) \\
(A \lor B) (a_0)
\]

\[
A (a_0) \\
(\overline{B} \lor D) (a_0) \\
\overline{A} (a_0)
\]

\[ \times \]

\[
B (a_0) \\
(\overline{B} \lor D) (a_0) \\
\overline{A} (a_0)
\]

\[ \times \]

\[
\overline{B} (a_0) \\
D (a_0)
\]

\[ \checkmark \]
Previous example in a picture

\[(A \sqcup B) \cap ((\neg B \sqcup D) \cap \neg A) (a_0)\]
\[((\neg B \sqcup D) \cap \neg A) (a_0)\]
\[(A \sqcup B) (a_0)\]

\[A (a_0)\]
\[\neg B (a_0)\]
\[\neg A (a_0)\]

\[B (a_0)\]
\[\neg B (a_0)\]
\[\neg A (a_0)\]

\[\neg B (a_0)\]
\[D (a_0)\]

\[\neg A (a_0)\]

\[A_3\]
Previous example in a picture

\[
(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A) (a_0)
\]

\[
((\neg B \sqcup D) \sqcap \neg A) (a_0)
\]

\[
(A \sqcup B) (a_0)
\]

\[
A (a_0)
\]

\[
(\neg B \sqcup D) (a_0)
\]

\[
\neg A (a_0)
\]

\[
\neg B (a_0)
\]

\[
D (a_0)
\]

\[
A_4
\]
Tableau example: only □ and ⊓

In our example, we had the complete and clash-free ABox $\mathcal{A}_4$:

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)(a_0) \quad (A \sqcup B)(a_0)$$

$$((\neg B \sqcup D) \sqcap \neg A)(a_0) \quad B(a_0) \quad (\neg B \sqcup D)(a_0) \quad \neg A(a_0) \quad D(a_0)$$

Can build from $\mathcal{A}_4$ the interpretation $\mathcal{I}$ with:

- $\Delta^\mathcal{I} = \{a_0\}$ use individuals from $\mathcal{A}_4$
- $A^\mathcal{I} = \emptyset$ since $\mathcal{A}_4$ does not contain $A(a_0)$
- $B^\mathcal{I} = D^\mathcal{I} = \{a_0\}$ since $\mathcal{A}_4$ contains $B(a_0)$ and $D(a_0)$

We can verify that $(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)^\mathcal{I} = \{a_0\}$.

- $\mathcal{I}$ witnesses the satisfiability of $C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)$
Tableau example: $\forall$ and $\exists$

Let's use the tableau procedure to test satisfiability of

$$C_0 = \exists R. A \sqcap \forall R. \neg A$$
Let’s use the tableau procedure to test satisfiability of 

\[ C_0 = \exists R. A \sqcap \forall R. \neg A \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ (\exists R. A \sqcap \forall R. \neg A)(a_0) \} \).
Let's use the tableau procedure to test satisfiability of

\[ C_0 = \exists R. A \land \forall R. \neg A \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ (\exists R. A \land \forall R. \neg A)(a_0) \} \).

Apply \( \land \)-rule to \( A_0 \):

get \( S = \{ A'_0 \} \) where \( A'_0 = A_0 \cup \{ (\exists R. A(a_0), (\forall R. \neg A)(a_0) \} \).
Let’s use the tableau procedure to test satisfiability of

\[ C_0 = \exists R.A \cap \forall R.\neg A \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{(\exists R.A \cap \forall R.\neg A)(a_0)\} \).

**Apply \( \cap \)-rule to \( A_0 \):**
get \( S = \{ A'_0 \} \) where \( A'_0 = A_0 \cup \{(\exists R.A)(a_0), (\forall R.\neg A)(a_0)\} \).
Tableau example: $\forall$ and $\exists$

Let's use the tableau procedure to test satisfiability of

$$C_0 = \exists R. A \sqcap \forall R. \neg A$$

Start with $S = \{ A_0 \}$ where $A_0 = \{ (\exists R. A \sqcap \forall R. \neg A)(a_0) \}$.

Apply $\sqcap$-rule to $A_0$:

get $S = \{ A'_0 \}$ where $A'_0 = A_0 \cup \{ (\exists R. A)(a_0), (\forall R. \neg A)(a_0) \}$.

Apply $\exists$-rule to $A'_0$:

get $S = \{ A''_0 \}$ where $A''_0 = A'_0 \cup \{ R(a_0, a_1), A(a_1) \}$. 
Let’s use the tableau procedure to test satisfiability of

\[ C_0 = \exists R. A \sqcap \forall R. \neg A \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ ( \exists R. A \sqcap \forall R. \neg A ) (a_0) \} \).

**Apply \( \sqcap \)-rule to \( A_0 \):**
get \( S = \{ A'_0 \} \) where \( A'_0 = A_0 \cup \{ (\exists R. A)(a_0), (\forall R. \neg A)(a_0) \} \).

**Apply \( \exists \)-rule to \( A'_0 \):**
get \( S = \{ A''_0 \} \) where \( A''_0 = A'_0 \cup \{ R(a_0, a_1), A(a_1) \} \).
Let’s use the tableau procedure to test satisfiability of

\[ C_0 = \exists R.A \sqcap \forall R.\neg A \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ (\exists R.A \sqcap \forall R.\neg A)(a_0) \} \).

**Apply \( \sqcap \)-rule to \( A_0 \):**
get \( S = \{ A'_0 \} \) where \( A'_0 = A_0 \cup \{ (\exists R.A)(a_0), (\forall R.\neg A)(a_0) \} \).

**Apply \( \exists \)-rule to \( A'_0 \):**
get \( S = \{ A''_0 \} \) where \( A''_0 = A'_0 \cup \{ R(a_0, a_1), A(a_1) \} \).

**Apply \( \forall \)-rule to \( A''_0 \):**
get \( S = \{ A'''_0 \} \) where \( A'''_0 = A''_0 \cup \{ \neg A(a_1) \} \).
Let’s use the tableau procedure to test satisfiability of

\[ C_0 = \exists R. A \sqcap \forall R. \neg A \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ (\exists R. A \sqcap \forall R. \neg A)(a_0) \} \).

**Apply \( \sqcap \)-rule to \( A_0 \):**
get \( S = \{ A'_0 \} \) where \( A'_0 = A_0 \cup \{(\exists R. A)(a_0), (\forall R. \neg A)(a_0)\} \).

**Apply \( \exists \)-rule to \( A'_0 \):**
get \( S = \{ A''_0 \} \) where \( A''_0 = A'_0 \cup \{R(a_0, a_1), A(a_1)\} \).

**Apply \( \forall \)-rule to \( A''_0 \):**
get \( S = \{ A'''_0 \} \) where \( A'''_0 = A''_0 \cup \{\neg A(a_1)\} \).

\( A'''_0 \) contains clash \( \{A(a_1), \neg A(a_1)\} \)!
Let’s use the tableau procedure to test satisfiability of

\[ C_0 = \exists R. A \land \forall R. \neg A \]

Start with \( S = \{ A_0 \} \) where \( A_0 = \{ (\exists R. A \land \forall R. \neg A)(a_0) \} \).

**Apply \( \land \)-rule to \( A_0 \):**
get \( S = \{ A'_0 \} \) where \( A'_0 = A_0 \cup \{(\exists R. A)(a_0), (\forall R. \neg A)(a_0)\} \).

**Apply \( \exists \)-rule to \( A'_0 \):**
get \( S = \{ A''_0 \} \) where \( A''_0 = A'_0 \cup \{ R(a_0, a_1), A(a_1) \} \).

**Apply \( \forall \)-rule to \( A''_0 \):**
get \( S = \{ A'''_0 \} \) where \( A'''_0 = A''_0 \cup \{ \neg A(a_1) \} \).

The only set in \( S \) contains a clash \( \Rightarrow C_0 \) is unsatisfiable
Previous example in a picture

\[(\exists R. A \sqcap \forall R. \neg A)(a_0)\]
Previous example in a picture

\[(\exists R. A \sqcap \forall R. \neg A) (a_0)\]
\[(\exists R. A) (a_0)\]
\[(\forall R. \neg A) (a_0)\]

\[\neg\text{-rule}\]
(∃R.A \land \forall R.\lnot A) (a_0)
(∃R.A) (a_0)
(∀R.\lnot A) (a_0)
R (a_0, a_1) \quad \exists\text{-rule}
A (a_1)
\[(\exists R. A \sqcap \forall R. \neg A) (a_0)\]

\[(\exists R. A) (a_0)\]

\[(\forall R. \neg A) (a_0)\]

\[R (a_0, a_1)\]

\[A (a_1) \quad \forall\text{-rule}\]

\[\neg A (a_1)\]
Previous example in a picture

\[
(\exists R.A \land \forall R.\neg A)(a_0)
\]

\[
(\exists R.A)(a_0)
\]

\[
(\forall R.\neg A)(a_0)
\]

\[
R(a_0, a_1)
\]

\[
A(a_1)
\]

\[
\neg A(a_1)
\]

✘
Tableau example: $\forall$ and $\exists$

Suppose that we consider a slightly different concept

$$C_0 = \exists R.A \cap \forall R.\neg B$$

Now the tableau algorithm yields the following complete, clash-free ABox:

$$(\exists R.A \cap \forall R.\neg B)(a_0) \quad (\exists R.A)(a_0) \quad (\forall R.\neg B)(a_0) \quad R(a_0, a_1) \quad A(a_1) \quad \neg B(a_1)$$
Suppose that we consider a slightly different concept

\[ C_0 = \exists R.A \sqcap \forall R.\neg B \]

Now the tableau algorithm yields the following complete, clash-free ABox:

\[ (\exists R.A \sqcap \forall R.\neg B)(a_0) \ (\exists R.A)(a_0) \ (\forall R.\neg B)(a_0) \ R(a_0, a_1) \ A(a_1) \ \neg B(a_1) \]

Corresponding interpretation \( \mathcal{I} \):

- \( \Delta^\mathcal{I} = \{a_0, a_1\} \)
- \( A^\mathcal{I} = \{a_1\} \)
- \( B^\mathcal{I} = \emptyset \)
- \( R^\mathcal{I} = \{(a_0, a_1)\} \)

Can check that \( \mathcal{I} \) is such that \( C_0^\mathcal{I} = \{a_0\} \).
Exercise: Concept satisfiability via tableau

Use the tableau algorithm to decide which of the following concepts is satisfiable:

1. \( C_1 = (\exists R. (A \cap B)) \cap (\forall R. (\neg A \cup D)) \)
2. \( C_2 = (\exists R. \exists S. A) \cap (\forall R. \forall S. \neg A) \)
3. \( C_3 = (\exists R. B) \cap (\forall R. (\forall R. A)) \cap (\forall R. \neg A) \)

If a concept is found to be satisfiable, use the result to construct an interpretation in which the concept is non-empty.
Let’s call our tableau algorithm CSat (for concept satisfiability). To show that CSat is a decision procedure, we must show:

**Termination**: The algorithm CSat always terminates.

**Soundness**: CSat outputs “yes” on input $C_0 \Rightarrow C_0$ is satisfiable.

**Completeness**: $C_0$ satisfiable $\Rightarrow$ CSat will output “yes”.
Preliminary definitions

Subconcepts of a concept:

\[
\begin{align*}
\text{sub}(\top) &= \{\top\} \\
\text{sub}(\bot) &= \{\bot\} \\
\text{sub}(A) &= \{A\} \\
\text{sub}(\neg C) &= \{\neg C\} \cup \text{sub}(C) \\
\text{sub}(\exists R.C) &= \{\exists R.C\} \cup \text{sub}(C) \\
\text{sub}(\forall R.C) &= \{\forall R.C\} \cup \text{sub}(C) \\
\text{sub}(C_1 \sqcup C_2) &= \{C_1 \sqcup C_2\} \cup \text{sub}(C_1) \cup \text{sub}(C_2) \\
\text{sub}(C_1 \sqcap C_2) &= \{C_1 \sqcap C_2\} \cup \text{sub}(C_1) \cup \text{sub}(C_2)
\end{align*}
\]
Preliminary definitions

Subconcepts of a concept:

\[
\begin{align*}
\text{sub}(\top) &= \{\top\} \\
\text{sub}(\bot) &= \{\bot\} \\
\text{sub}(A) &= \{A\} \\
\text{sub}(\neg C) &= \{\neg C\} \cup \text{sub}(C) \\
\text{sub}(\exists R. C) &= \{\exists R. C\} \cup \text{sub}(C) \\
\text{sub}(\forall R. C) &= \{\forall R. C\} \cup \text{sub}(C) \\
\text{sub}(C_1 \sqcup C_2) &= \{C_1 \sqcup C_2\} \cup \text{sub}(C_1) \cup \text{sub}(C_2) \\
\text{sub}(C_1 \sqcap C_2) &= \{C_1 \sqcap C_2\} \cup \text{sub}(C_1) \cup \text{sub}(C_2)
\end{align*}
\]

Role depth of a concept:

\[
\begin{align*}
\text{depth}(A) &= \text{depth}(\top) = \text{depth}(\bot) = 0 \\
\text{depth}(\neg C) &= \text{depth}(C) \\
\text{depth}(\exists R. C) &= \text{depth}(\forall R. C) = \text{depth}(C) + 1 \\
\text{depth}(C_1 \sqcup C_2) &= \text{depth}(C_1 \sqcap C_2) = \max(\text{depth}(C_1), \text{depth}(C_2))
\end{align*}
\]
## Preliminary definitions

### Subconcepts of a concept:

\[ \text{sub}(\mathcal{C}) \leq |\mathcal{C}| \]

\[
\begin{align*}
\text{sub}(\top) &= \{\top\} \\
\text{sub}(\bot) &= \{\bot\} \\
\text{sub}(A) &= \{A\} \\
\text{sub}(\neg C) &= \{\neg C\} \cup \text{sub}(C) \\
\text{sub}(\exists R.C) &= \{\exists R.C\} \cup \text{sub}(C) \\
\text{sub}(\forall R.C) &= \{\forall R.C\} \cup \text{sub}(C) \\
\text{sub}(C_1 \sqcup C_2) &= \{C_1 \sqcup C_2\} \cup \text{sub}(C_1) \cup \text{sub}(C_2) \\
\text{sub}(C_1 \sqcap C_2) &= \{C_1 \sqcap C_2\} \cup \text{sub}(C_1) \cup \text{sub}(C_2)
\end{align*}
\]

### Role depth of a concept:

\[ \text{depth}(\mathcal{C}) \leq |\mathcal{C}| \]

\[
\begin{align*}
\text{depth}(A) &= \text{depth}(\top) = \text{depth}(\bot) = 0 \\
\text{depth}(\neg C) &= \text{depth}(C) \\
\text{depth}(\exists R.C) &= \text{depth}(\forall R.C) = \text{depth}(C) + 1 \\
\text{depth}(C_1 \sqcup C_2) &= \text{depth}(C_1 \sqcap C_2) = \max(\text{depth}(C_1), \text{depth}(C_2))
\end{align*}
\]
Termination of CSat

Suppose we run CSat starting from $S = \{\{C_0(a_0)\}\}$. 

1. If $D(b) \in A$, then $D \in \text{sub}(C_0)$.
2. The set of role assertions in $A$ forms a tree.
3. If $D(b) \in A$ and the unique path from $a_0$ to $b$ has length $k$, then $\text{depth}(D) \leq \text{depth}(C_0) - k$.
4. For every individual $b$ in $A$, there are at most $|C_0|$ individuals $c$ such that $R(b,c) \in A$ (at most one per existential concept).

Thus: bound on the size of ABoxes generated by the procedure.

The tableau procedure only adds assertions to ABoxes, so eventually all ABoxes will contain a clash or will be complete.
Termination of CSat

Suppose we run CSat starting from $S = \{\{C_0(a_0)\}\}$.

We observe that for every ABox $A$ generated by the procedure:

1. if $D(b) \in A$, then $D \in \text{sub}(C_0) \Rightarrow A$ contains at most $|C_0|$ concept assertions per individual
2. the set of role assertions in $A$ forms a tree
3. if $D(b) \in A$ and the unique path from $a_0$ to $b$ has length $k$, then $\text{depth}(D) \leq \text{depth}(C_0) - k \Rightarrow$ each individual in $A$ is at distance $\leq \text{depth}(C_0)$ from $a_0$
4. for every individual $b$ in $A$, there are at most $|C_0|$ individuals $c$ such that $R(b, c) \in A$ for some $R$ (at most one per existential concept)

Thus:

- bound on the size of ABoxes generated by the procedure.
- The tableau procedure only adds assertions to ABoxes $\Rightarrow$ eventually all ABoxes will contain a clash or will be complete.
Suppose we run CSat starting from $S = \{\{C_0(a_0)\}\}$.

We observe that for every ABox $\mathcal{A}$ generated by the procedure:

1. if $D(b) \in \mathcal{A}$, then $D \in \text{sub}(C_0)$
Termination of CSat

Suppose we run CSat starting from \( S = \{ \{ C_0(a_0) \} \} \).

We observe that for every ABox \( A \) generated by the procedure:

1. if \( D(b) \in A \), then \( D \in \text{sub}(C_0) \)
   - \( A \) contains at most \( |C_0| \) concept assertions per individual
Termination of CSat

Suppose we run CSat starting from $S = \{\{C_0(a_0)\}\}$.

We observe that for every ABox $\mathcal{A}$ generated by the procedure:

1. if $D(b) \in \mathcal{A}$, then $D \in \text{sub}(C_0)$
   - $\mathcal{A}$ contains at most $|C_0|$ concept assertions per individual

2. the set of role assertions in $\mathcal{A}$ forms a tree
Termination of CSat

Suppose we run CSat starting from $S = \{\{C_0(a_0)\}\}$.

We observe that for every ABox $\mathcal{A}$ generated by the procedure:

1. if $D(b) \in \mathcal{A}$, then $D \in \text{sub}(C_0)$
   - $\mathcal{A}$ contains at most $|C_0|$ concept assertions per individual

2. the set of role assertions in $\mathcal{A}$ forms a tree

3. if $D(b) \in \mathcal{A}$ and the unique path from $a_0$ to $b$ has length $k$, then $\text{depth}(D) \leq \text{depth}(C_0) - k$

Thus: bound on the size of ABoxes generated by the procedure.

The tableau procedure only adds assertions to ABoxes $\Rightarrow$ eventually all ABoxes will contain a clash or will be complete.
Termination of CSat

Suppose we run CSat starting from $S = \{ \{ C_0(a_0) \} \}$.

We observe that for every ABox $\mathcal{A}$ generated by the procedure:

1. if $D(b) \in \mathcal{A}$, then $D \in \text{sub}(C_0)$
   
   $\Rightarrow$ $\mathcal{A}$ contains at most $|C_0|$ concept assertions per individual

2. the set of role assertions in $\mathcal{A}$ forms a tree

3. if $D(b) \in \mathcal{A}$ and the unique path from $a_0$ to $b$ has length $k$,
   then depth$(D) \leq$ depth$(C_0) - k$

   $\Rightarrow$ each individual in $\mathcal{A}$ is at distance $\leq$ depth$(C_0)$ from $a_0$
Termination of CSat

Suppose we run CSat starting from $S = \{\{C_0(a_0)\}\}$.

We observe that for every ABox $A$ generated by the procedure:

1. if $D(b) \in A$, then $D \in \text{sub}(C_0)$
   - $A$ contains at most $|C_0|$ concept assertions per individual

2. the set of role assertions in $A$ forms a tree

3. if $D(b) \in A$ and the unique path from $a_0$ to $b$ has length $k$, then $\text{depth}(D) \leq \text{depth}(C_0) - k$
   - each individual in $A$ is at distance $\leq \text{depth}(C_0)$ from $a_0$

4. for every individual $b$ in $A$, there are at most $|C_0|$ individuals $c$ such that $R(b, c) \in A$ for some $R$ (at most one per existential concept)
Termination of CSat

Suppose we run CSat starting from $S = \{\{C_0(a_0)\}\}$.

We observe that for every ABox $A$ generated by the procedure:

1. if $D(b) \in A$, then $D \in \text{sub}(C_0)$
   - $A$ contains at most $|C_0|$ concept assertions per individual

2. the set of role assertions in $A$ forms a tree

3. if $D(b) \in A$ and the unique path from $a_0$ to $b$ has length $k$,
   then $\text{depth}(D) \leq \text{depth}(C_0) - k$
   - each individual in $A$ is at distance $\leq \text{depth}(C_0)$ from $a_0$

4. for every individual $b$ in $A$, there are at most $|C_0|$ individuals $c$ such that $R(b, c) \in A$ for some $R$ (at most one per existential concept)

Thus: bound on the size of ABoxes generated by the procedure.
**Termination of CSat**

Suppose we run CSat starting from $S = \{\{C_0(a_0)\}\}$.

We observe that for every ABox $A$ generated by the procedure:

1. if $D(b) \in A$, then $D \in \text{sub}(C_0)$
   ▶ $A$ contains at most $|C_0|$ concept assertions per individual

2. the set of role assertions in $A$ forms a tree

3. if $D(b) \in A$ and the unique path from $a_0$ to $b$ has length $k$, then $\text{depth}(D) \leq \text{depth}(C_0) - k$
   ▶ each individual in $A$ is at distance $\leq \text{depth}(C_0)$ from $a_0$

4. for every individual $b$ in $A$, there are at most $|C_0|$ individuals $c$ such that $R(b, c) \in A$ for some $R$ (at most one per existential concept)

Thus: **bound on the size of ABoxes** generated by the procedure.

The tableau procedure **only adds assertions** to ABoxes
⇒ **eventually all ABoxes will contain a clash or will be complete**
Soundness of $\text{CSat}$ (1)

Suppose that $\text{CSat}$ returns “yes” on input $C_0$.

Then $S$ must contain a complete and clash-free ABox $A$. 
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Suppose that CSat returns “yes” on input $C_0$.

Then $S$ must contain a complete and clash-free ABox $A$.

Use $A$ to define an interpretation $I$ as follows:

- $\Delta^I = \{a \mid a$ is an individual in $A\}$
- $A^I = \{a \mid A(a) \in A\}$
- $R^I = \{(a, b) \mid R(a, b) \in A\}$

Claim: $I$ is such that $C^I_0 \neq \emptyset$
Suppose that CSat returns “yes” on input $C_0$.

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Use $A$ to define an interpretation $\mathcal{I}$ as follows:

- $\Delta^{\mathcal{I}} = \{ a \mid a \text{ is an individual in } A \}$
- $A^{\mathcal{I}} = \{ a \mid A(a) \in A \}$
- $R^{\mathcal{I}} = \{ (a, b) \mid R(a, b) \in A \}$

Claim: $\mathcal{I}$ is such that $C_0^{\mathcal{I}} \neq \emptyset$

To show the claim, we prove by induction on the size of concepts that:

$$D(b) \in A \Rightarrow b \in D^{\mathcal{I}}$$
Soundness of CSat (2)

**Base case:** \(D = A\) or \(D = \neg A\) or \(D = \top\) or \(D = \bot\)

- If \(D = A\), then \(b \in A\).
- If \(D = \neg A\), then \(A(b) \notin A\), so \(b \in \neg A\).
- If \(D = \top\), trivially \(b \in \top\).
- We cannot have \(D = \bot\) since clash-free.

**Induction hypothesis (IH):** suppose statement holds whenever \(|D| \leq k\).

**Induction step:** show statement holds for \(D\) with \(|D| = k + 1\).

Again, many cases to consider:

- \(D = E \odot F\): since \(A\) is complete, it must contain both \(E(b)\) and \(F(b)\).
  - Applying the IH, we get \(b \in E\) and \(b \in F\), hence \(b \in (E \odot F)\).

- \(D = \exists R. E\): since \(A\) is complete, there is some \(c\) such that \(R(b, c) \in A\) and \(E(c) \in A\). Then \((b, c) \in R\) and by the IH, we get \(c \in E\), so \(b \in (\exists R. E)\).

- \(D = E \uplus F\): left as practice.

- \(D = \forall R. E\): left as practice.
Soundness of CSat (2)

**Base case:** $D = A$ or $D = \neg A$ or $D = \top$ or $D = \bot$

If $D = A$, then $b \in A^\mathcal{I}$.

If $D = \neg A$, then $A(b) \notin A$, so $b \in \neg A^\mathcal{I}$.

If $D = \top$, trivially $b \in \top^\mathcal{I} = \Delta^\mathcal{I}$. We cannot have $D = \bot$ since clash-free.

**Induction hypothesis (IH):** suppose statement holds whenever $|D| \leq k$

**Induction step:** show statement holds for $D$ with $|D| = k + 1$

Again, many cases to consider:

- **$D = E \cap F$:** since $\mathcal{A}$ is complete, it must contain both $E(b)$ and $F(b)$. Applying the IH, we get $b \in E^\mathcal{I}$ and $b \in F^\mathcal{I}$, hence $b \in (E \cap F)^\mathcal{I}$

- **$D = \exists R.E$:** since $\mathcal{A}$ is complete, there is some $c$ such that $R(b, c) \in \mathcal{A}$ and $E(c) \in \mathcal{A}$. Then $(b, c) \in R^\mathcal{I}$ and by the IH, we get $c \in E^\mathcal{I}$, so $b \in (\exists R.E)^\mathcal{I}$

- **$D = E \cup F$:** left as practice

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Suppose that the concept $C_0$ is satisfiable.

Then the ABox $\{C_0(a_0)\}$ must be satisfiable too.
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We observe that the tableau rules are *satisfiability-preserving*:

- If an ABox $\mathcal{A}$ is satisfiable and $\mathcal{A}'$ is the result of applying a rule to $\mathcal{A}$, then $\mathcal{A}'$ is also satisfiable.

- If an ABox $\mathcal{A}$ is satisfiable and $\mathcal{A}_1$ and $\mathcal{A}_2$ are obtained when applying a rule to $\mathcal{A}$, then either $\mathcal{A}_1$ is satisfiable or $\mathcal{A}_2$ is satisfiable.
Completeness of CSat

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We start with a satisfiable ABox and the rules are satisfiability-preserving, so eventually we will reach a complete, satisfiable (thus: clash-free) ABox.
Complexity of CSat

Bad news: our algorithm may require exponential time and space...

To see why, consider what happens if we run CSat on the concept

\[
\bigcap_{0 \leq i < n} \forall R_1 \ldots \forall R_i \left( \exists R.B \sqcap \exists R.\neg B \right)
\]

i times
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\]

\( \text{i times} \)

Good news: can modify the procedure so it runs in polynomial space

- instead of a set of ABoxes, keep only one ABox in memory at a time
  - when apply the \( \sqcup \)-rule, first examine \( A_1 \), then afterwards examine \( A_2 \)
  - remember that second disjunct stills needs to be checked

- explore the children of an individual one at a time
  - possible because no interaction between the different “branches”
  - store which \( \exists R.C \) concepts have been tested, which are left to do

- this allows us to keep at most \( |C_0| \) individuals in memory at a time
Complexity of $\mathcal{ALC}$ concept satisfiability

Hierarchy of complexity classes

$\text{PTIME} \subseteq \text{NP} \subseteq \ldots \subseteq \text{PSPACE} \subseteq \text{EXPTIME} \subseteq \text{NEXPTIME} \subseteq \ldots \subseteq \text{EXPSPACE} \ldots$

(it is believed that all inclusions are strict)
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PSPACE-complete problems = hardest problems in PSPACE
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PSPACE = class of decision problems solvable using polynomial space

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**Theorem**: $\mathcal{ALC}$ concept satisfiability (no TBox) is PSPACE-complete.

- Membership in PSPACE shown using modified tableau procedure
- Hardness for PSPACE shown by giving a reduction from some known PSPACE-hard problem (e.g. QBF validity)
Extending the tableau algorithm to KBs
Now we want to modify the algorithm to handle KB satisfiability.

Adding an ABox is easy: simply start with \( \{A\} \) instead of \( \{C_0(a_0)\} \)
Now we want to modify the algorithm to handle KB satisfiability.

Adding an ABox is easy: simply start with \( \{ \mathcal{A} \} \) instead of \( \{ C_0(a_0) \} \).

Adding a TBox is a bit more tricky...

Idea: if we have \( C \sqsubseteq D \), then every element must satisfy either \( \neg C \) or \( D \).
Extension to KB satisfiability

Now we want to modify the algorithm to handle KB satisfiability.

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Adding a TBox is a bit more tricky...

Idea: if we have \( C \sqsubseteq D \), then every element must satisfy either \( \neg C \) or \( D \)

Concretely, we might try adding the following rule:

**TBox rule:** if \( a \) is in \( \mathcal{A} \), \( C \sqsubseteq D \in \mathcal{T} \), & \( \text{NNF}(\neg C) \cup \text{NNF}(D) \) (\( a \) \( \notin \mathcal{A} \)) then replace \( \mathcal{A} \) with \( \mathcal{A} \cup \{(\text{NNF}(\neg C) \cup \text{NNF}(D))(a)\} \)
Examples: KB satisfiability

Let’s try the modified procedure on the KB\( (\mathcal{T}, \{A(a)\}) \) where

\[
\mathcal{T} = \{ A \sqsubseteq \exists R. B \quad B \sqsubseteq D \quad \exists R. D \sqsubseteq \neg A \} 
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Now try it on the KB \((\{F \sqsubseteq \exists S.F\}, \{F(a)\})\).
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Now try it on the KB \((\{F \sqsubseteq \exists S. F\}, \{F(a)\})\).

Seems we have a problem... How can we ensure termination?
Blocking

**Basic idea:** if two individuals “look the same”, then it is unnecessary to explore both of them
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Formally, give individuals $a, b$ in $\mathcal{A}$, we say that $b$ blocks $a$ if:

- $\{ C \mid C(a) \in \mathcal{A} \} \subseteq \{ C \mid C(b) \in \mathcal{A} \}$

- $b$ was present in $\mathcal{A}$ before $a$ was introduced

Say that individual $a$ is blocked (in $\mathcal{A}$) if some $b$ blocks $a$. 
**Blocking**

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Say that individual \textbf{$a$ is blocked} (in $\mathcal{A}$) if some $b$ blocks $a$.

Modify rules so that they \textbf{only apply to unblocked individuals}.
Tableau rules for KBs

\(-\text{rule:}\) if \((C_1 \cap C_2)(a) \in \mathcal{A},\) \textit{a is not blocked,} and \(\{C_1(a), C_2(a)\} \not\subseteq \mathcal{A},\)
then replace \(\mathcal{A}\) with \(\mathcal{A} \cup \{C_1(a), C_2(a)\}\)

\(\square\text{-rule:}\) if \((C_1 \sqcup C_2)(a) \in \mathcal{A},\) \textit{a is not blocked,} and \(\{C_1(a), C_2(a)\} \cap \mathcal{A} = \emptyset,\)
then replace \(\mathcal{A}\) with \(\mathcal{A} \cup \{C_1(a)\}\) \textit{and} \(\mathcal{A} \cup \{C_2(a)\}\)

\(\forall\text{-rule:}\) if \(\{\forall R. C(a), R(a, b)\} \in \mathcal{A},\) \textit{a is not blocked,} and \(C(b) \not\in \mathcal{A},\)
then replace \(\mathcal{A}\) with \(\mathcal{A} \cup \{C(b)\}\)

\(\exists\text{-rule:}\) if \(\{\exists R. C(a)\} \in \mathcal{A},\) \textit{a is not blocked,} and no \(\{R(a, b), C(b)\} \subseteq \mathcal{A},\)
then \textit{pick a new individual name} \(d\) and replace \(\mathcal{A}\) with \(\mathcal{A} \cup \{R(a, d), C(d)\}\)

\(\subseteq\text{-rule:}\) if \(a\) appears in \(\mathcal{A}\) and \textit{a is not blocked,} \(C \sqsubseteq D \in \mathcal{T},\) and
\((\text{NNF}(\neg C) \sqcup \text{NNF}(D))(a) \not\in \mathcal{A},\)
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Example: blocking

Let’s try blocking on the problematic KB \( \{F \sqsubseteq \exists S.F, \{F(a)\}\} \).
Let’s try blocking on the problematic KB \( \{ F \sqsupseteq \exists S.F, \{ F(a) \} \} \).

\[
\begin{align*}
F(a_0) \\
(\neg F \sqcup \exists S.F)(a_0) \\
\neg F(a_0) & \quad (\exists S.F)(a_0) \\
\text{✘} & \quad S(a_0, a_1) \\
F(a_1) & \quad a_1 \text{ is blocked by } a_0
\end{align*}
\]
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Let’s try blocking on the problematic KB \( \{ F \sqsubseteq \exists S.F, \{ F(a) \} \} \).

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F(a_0) & \quad \text{(\( a_0 \) is blocked by \( a_0 \))} \\
(\neg F \sqcup \exists S.F)(a_0) & \\
\neg F(a_0) & \quad (\exists S.F)(a_0) \\
\times & \quad S(a_0, a_1) \\
& \quad F(a_1) \\
\end{align*}
\]

We obtain a complete and clash-free ABox \( \Rightarrow \) the KB is satisfiable!
Another blocking example

Consider the following TBox

\[ T = \{ A \sqsubseteq \exists R.A, A \sqsubseteq B, \exists R.B \sqsubseteq D \} \]

and suppose we want to test whether \( T \models A \sqsubseteq D \).

We can do this by running the algorithm on the KB \((T, \{(A \sqcap \neg D)(a_0)\})\).
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**Result:** the KB is unsatisfiable \( \Rightarrow T \models A \sqsubseteq D \)
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We can do this by running the algorithm on the KB \((T, \{(A \sqcap \neg D) (a_0)\})\).

Result: the KB is unsatisfiable \( \Rightarrow T \models A \sqsubseteq D \)

Observation: an individual can be blocked, then later become unblocked
Properties of the tableau algorithm

Let’s call our new tableau algorithm KBSat (for KB satisfiability).

**Termination**: The algorithm KBSat always terminates.
- similar to before: bound the size of generated ABoxes
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- again, we use complete, clash-free ABox to build a model
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So: KBSat is a decision procedure for KB satisfiability.
The KBSat algorithm generates ABoxes of at most exponential size.

But even if with the tricks from earlier, need exponential space.

- single branch may be exponentially long
Complexity of KB satisfiability

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This is most likely not optimal, as we can show the following:

**Theorem:** $ALC$ KB satisfiability is EXPTIME-complete.
Complexity of KB satisfiability

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This is most likely not optimal, as we can show the following:

**Theorem**: $\mathcal{ALC}$ KB satisfiability is EXPTIME-complete.

This result means **no polynomial-time algorithm can ever be found**.
Optimizations

Despite high worst-case complexity, tableau algorithms for $ALC$ and other expressive DLs can work well in practice.
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- explore only one branch of one ABox at a time
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- reduce number of $\sqcap$’s created by TBox inclusions (absorption)
- reduce number of satisfiability checks during classification
Absorption (1)

When $\mathcal{T} = \{ C_i \sqsubseteq D_i \mid 1 \leq i \leq n \}$, we get $n$ disjunctions per individual:

$$(\text{NNF}(\neg C_1) \sqcup \text{NNF}(D_1))(a), \ldots, (\text{NNF}(\neg C_n) \sqcup \text{NNF}(D_n))(a)$$
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**Observation**: if have inclusion $A \sqsubseteq D$ with $A$ atomic

- if don't have $A(a)$, can satisfy the inclusion by choosing $\neg A(a)$
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So for inclusions with atomic left-hand side, can replace $\sqsubseteq$-rule by:

$\sqsubseteq^{an}$-rule: if $A(a) \in \mathcal{A}$, $a$ is not blocked, $A \sqsubseteq D \in \mathcal{T}$ (with $A$ atomic), and $D(a) \not\in \mathcal{A}$, then replace $\mathcal{A}$ with $\mathcal{A} \cup \{D(a)\}$
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**Good news**: we’ve lowered the number of disjunctions!
Absorption (2)

**Second observation:** can transform some inclusions with complex concept on left into equivalent inclusions with atomic left-hand side.
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\[(A \cap C) \sqsubseteq D \quad \leadsto \quad A \sqsubseteq (\neg C \sqcup D)\]
Absorption (2)

**Second observation:** can transform some inclusions with complex concept on left into equivalent inclusions with atomic left-hand side

\[(A \cap C) \subseteq D \quad \sim \quad A \subseteq (\neg C \cup D)\]

**Absorption technique:**

1. preprocess the TBox by replacing inclusions with equivalent inclusions with atomic concept on left, whenever possible

2. when running tableau algorithm
   - use new $\sqsubseteq^at$-rule for inclusions $A \sqsubseteq D$ with $A$ atomic
   - use regular $\sqsubseteq$-rule for the other TBox inclusions
Example: Absorption

Let’s use absorption on the KB \((\mathcal{T}, \{A(a)\})\) from earlier with:

\[
\{ A \subseteq \exists R.B \quad B \subseteq D \quad \exists R.D \subseteq \neg A \}
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- first two inclusions in \( \mathcal{T} \) already have atomic concept on left
- third inclusion in \( \mathcal{T} \) can be equivalently written as \( A \sqsubseteq \forall R.\neg D \)
- so: only need to use \( \sqsubseteq^{at} \)-rule
Example: Absorption

Let’s use absorption on the KB \( (T, \{A(a)\}) \) from earlier with:

\[
\{ \quad A \sqsubseteq \exists R.B \quad B \sqsubseteq D \quad \exists R.D \sqsubseteq \lnot A \quad \}
\]

- first two inclusions in \( T \) already have atomic concept on left
- third inclusion in \( T \) can be equivalently written as \( A \sqsubseteq \forall R.\lnot D \)
- so: only need to use \( \sqsubseteq^{at} \)-rule

**Result**: completely avoid disjunction, algorithm terminates much faster!
Optimizations for classification

**Classification**: find all pairs of atomic concepts $A, B$ with $\mathcal{T} \models A \sqsubseteq B$

**Naïve approach**: test satisfiability of $A \sqcap \neg B$ w.r.t. $\mathcal{T}$ for all pairs $A, B$

- but $\mathcal{T}$ may contain hundreds or thousands of atomic concepts....
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**Naïve approach**: test satisfiability of $A \sqcap \neg B$ w.r.t. $\mathcal{T}$ for all pairs $A, B$

- but $\mathcal{T}$ may contain hundreds or thousands of atomic concepts....

Each satisfiability check is costly $\Rightarrow$ **want to reduce number of checks**
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Some ideas:

- some subsumptions are obvious
  - $A \sqsubseteq A$ and subsumptions that are explicitly stated in $\mathcal{T}$
- can use simple reasoning to obtain new (non-)subsumptions
  - if know $\mathcal{T} \models A \sqsubseteq B$ and $\mathcal{T} \models B \sqsubseteq D$, then $\mathcal{T} \models A \sqsubseteq D$
  - if know $\mathcal{T} \models A \sqsubseteq B$ and $\mathcal{T} \not\models A \sqsubseteq D$, then $\mathcal{T} \not\models B \sqsubseteq D$
Exercise: KB satisfiability via tableau

Consider the following TBox:

\[ T = \{ A \sqsubseteq \forall R.B, B \sqsubseteq \neg F, E \sqsubseteq G, A \sqsubseteq D \sqcup E, D \sqsubseteq \exists R.F, \exists R.\neg B \sqsubseteq G \} \]

Use the KBSat algorithm to decide whether:

1. \( T \models A \sqsubseteq E \)
2. \( T \models E \sqsubseteq G \)
3. \( T \models E \sqsubseteq F \)
4. \( T \models A \sqsubseteq G \)
5. \( T \models D \sqsubseteq G \)
6. \( T \models G \sqsubseteq F \)

You are encouraged to use the optimizations introduced in the course.