ONTOLOGIES & DESCRIPTION LOGICS

Parcours IA - Représentation des connaissances

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REASONING IN EXPRESSIVE DLS

TABLEAU METHOD

Tableau method: popular approach for reasoning in expressive DLs

- implemented in state-of-the-art DL reasoners

Tableau algorithms are used to decide satisfiability

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Idea: to determine whether a given (concept or KB) Ψ is satisfiable, try to construct a (representation of a) model of Ψ

- if we succeed, then we have shown that Ψ is satisfiable
- if we fail despite having considered all possibilities, then we have proven that Ψ is unsatisfiable

ALC CONCEPTS

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We say that an $\mathcal{ALC}$ concept C is in negation normal form (NNF) if the symbol $\neg$ only appears directly in front of atomic concepts.

- in NNF: $A \sqcap \neg B$, $\exists r.\neg A$, $\neg A \sqcup \neg B$
- not in NNF: $\neg(A \sqcap B)$, $\exists r.\neg(\forall s.B)$, $A \sqcup \neg \forall r.B$, $\neg \top$

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- not in NNF: $\neg(A \sqcap B)$, $\exists r.\neg(\forall s.B)$, $A \sqcup \neg \forall r.B$, $\neg\top$

Fact. Every $\mathcal{ALC}$ concept C can be transformed into an equivalent concept in NNF in linear time by applying the following rewrite rules:

$$\neg\top \rightsquigarrow \bot \qquad \neg(C \sqcap D) \rightsquigarrow \neg C \sqcup \neg D \qquad \neg(\forall r.C) \rightsquigarrow \exists r.\neg C$$
$$\neg\bot \rightsquigarrow \top \qquad \neg(C \sqcup D) \rightsquigarrow \neg C \sqcap \neg D \qquad \neg(\exists r.C) \rightsquigarrow \forall r.\neg C$$

Note: say C and D are equivalent if the empty TBox entails $C \equiv D$.

SATISFIABILITY OF ALC-CONCEPTS VIA TABLEAU

We begin by presenting a tableau algorithm for deciding satisfiability of $\mathcal{ALC}$-concepts (in NNF) w.r.t. the empty TBox

Procedure for testing satisfiability of C_0:

- We work with a set S of ABoxes
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- Stop applying rules when either:
  - every $\mathcal{A} \in S$ contains a clash, i.e. an assertion $\bot(b)$ or a pair of assertions $\{B(b), \neg B(b)\}$
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- Return 'yes, satisfiable' if some $\mathcal{A} \in S$ is clash-free, else "no".

$\sqcap$-rule: if $(C_1 \sqcap C_2)(a) \in \mathcal{A}$ and $\{C_1(a), C_2(a)\} \not\subseteq \mathcal{A}$
then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a), C_2(a)\}$

TABLEAU RULES FOR ALC

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$\exists$-rule: if $\{\exists r.C(a)\} \in \mathcal{A}$ and no b with $\{r(a, b), C(b)\} \subseteq \mathcal{A}$,
then **pick a new individual name** d and
replace $\mathcal{A}$ with $\mathcal{A} \cup \{r(a, d), C(d)\}$

FIRST EXAMPLE: $\sqcap$ AND $\sqcup$

Test satisfiability of concept $C = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)$

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Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ ((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A))(a_0) \}$.

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$\mathcal{A}'_1$ **contains clash** $\{A(a_0), \neg A(a_0)\}$**!**

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Apply $\sqcup$-rule to $\mathcal{A}'_2$:
get $S = \{ \mathcal{A}'_1, \mathcal{A}_3, \mathcal{A}_4 \}$ where $\mathcal{A}_3 = \mathcal{A}'_2 \cup \{\neg B(a_0)\}$, $\mathcal{A}_4 = \mathcal{A}'_2 \cup \{D(a_0)\}$

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$\mathcal{A}_4$ is complete, so we can stop.

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$\mathcal{A}_4$ is complete and contains no clash $\Rightarrow$ C_0 is satisfiable

PREVIOUS EXAMPLE IN GRAPHICAL FORMAT

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$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

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$$(A \sqcup B)\,(a_0)$$

PREVIOUS EXAMPLE IN GRAPHICAL FORMAT

Test satisfiability of concept $C = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)$

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$
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$A\,(a_0)$ | $B\,(a_0)$ **$\sqcup$-rule**

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$\sqcap$-**rule**

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$$\neg A\,(a_0)$$

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$A\,(a_0)$	$B\,(a_0)$	
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$A\,(a_0)$ | $B\,(a_0)$

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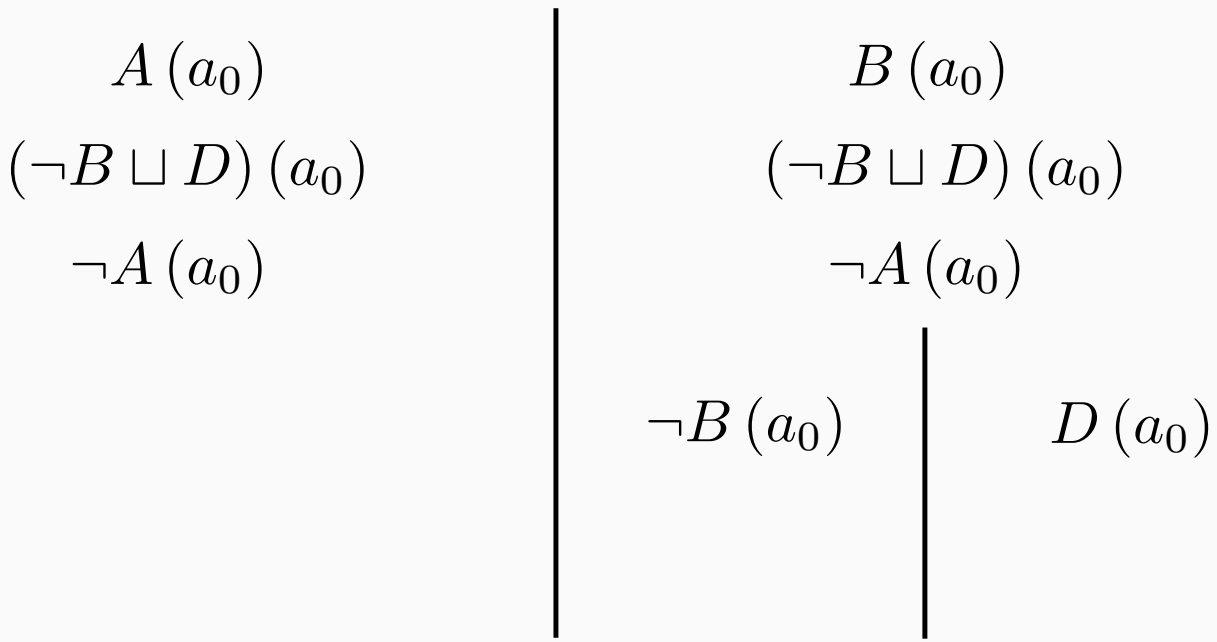

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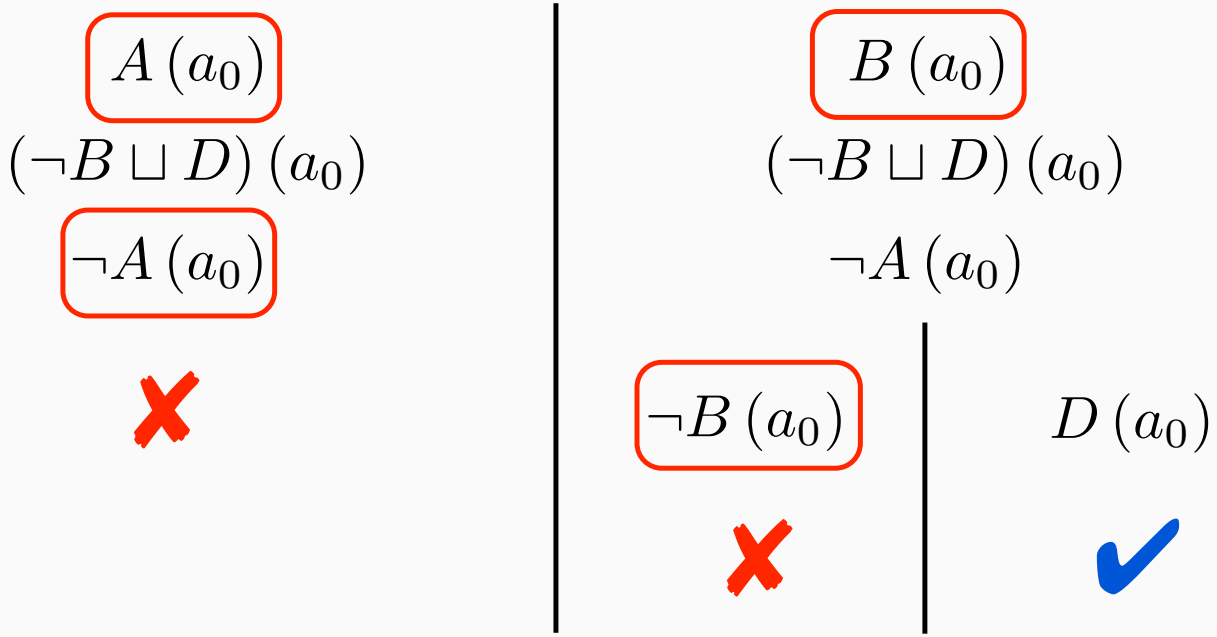

EXAMPLE: WITNESSING INTERPRETATION

In our example, we had the **complete and clash-free ABox** $\mathcal{A}_4$:

$$((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A))(a_0) \quad (A \sqcup B)(a_0)$$
$$((\neg B \sqcup D) \sqcap \neg A)(a_0) \quad B(a_0) \quad (\neg B \sqcup D)(a_0) \quad \neg A(a_0) \quad D(a_0)$$

Can build from $\mathcal{A}_4$ the interpretation $\mathcal{I}$ with:

- $\Delta^{\mathcal{I}} = \{a_0\}$ use individuals from $\mathcal{A}_4$
- $A^{\mathcal{I}} = \emptyset$ since $\mathcal{A}_4$ does not contain $A(a_0)$
- $B^{\mathcal{I}} = D^{\mathcal{I}} = \{a_0\}$ since $\mathcal{A}_4$ contains $B(a_0)$ and $D(a_0)$

We can verify that $(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)^{\mathcal{I}} = \{a_0\}$.

- **$\mathcal{I}$ witnesses the satisfiability of** $C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)$

ANOTHER EXAMPLE: $\forall$ AND $\exists$

Let's use the tableau procedure to test satisfiability of

$$C = \exists r.A \sqcap \forall r.\neg A$$

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Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ (\exists r.A \sqcap \forall r.\neg A)(a_0) \}$.

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Apply $\sqcap$-rule to $\mathcal{A}_0$:
get $S = \{ \mathcal{A}'_0 \}$ where $\mathcal{A}'_0 = \mathcal{A}_0 \cup \{(\exists r.A)(a_0), (\forall r.\neg A)(a_0)\}$.

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Apply $\exists$-rule to $\mathcal{A}_0'$:
get $S = \{ \mathcal{A}_0'' \}$ where $\mathcal{A}_0'' = \mathcal{A}_0' \cup \{r(a_0, a_1), A(a_1)\}$.

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Apply $\forall$-rule to $\mathcal{A}_0''$:
get $S = \{ \mathcal{A}_0''' \}$ where $\mathcal{A}_0''' = \mathcal{A}_0'' \cup \{\neg A(a_1)\}$.

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Apply $\forall$-rule to $\mathcal{A}_0''$:
get $S = \{ \mathcal{A}_0''' \}$ where $\mathcal{A}_0''' = \mathcal{A}_0'' \cup \{\neg A(a_1)\}$.

$\mathcal{A}_0'''$ **contains clash** $\{A(a_1), \neg A(a_1)\}$**!**

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The only ABox in S contains a clash $\Rightarrow$ C_0 is unsatisfiable

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Test satisfiability of concept $C = \exists r.A \sqcap \forall r.\neg A$

$$(\exists r.A \sqcap \forall r.\neg A)(a_0)$$

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Conclude that C is unsatisfiable

FURTHER EXAMPLE WITH $\forall$ AND $\exists$

Suppose that we consider a slightly different concept

$$C_0 = \exists r.A \sqcap \forall r.\neg \mathrm{B}$$

Now the algorithm yields the following complete, clash-free ABox:

$$(\exists r.A \sqcap \forall r.\neg B)(a_0) \quad (\exists r.A)(a_0) \quad (\forall r.\neg B)(a_0) \quad r(a_0, a_1) \quad A(a_1) \quad \neg B(a_1)$$

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Corresponding interpretation $\mathcal{I}$:

- $\Delta^{\mathcal{I}} = \{a_0, a_1\}$
- $A^{\mathcal{I}} = \{a_1\}$
- $B^{\mathcal{I}} = \emptyset$
- $r^{\mathcal{I}} = \{(a_0, a_1)\}$

Can check that $\mathcal{I}$ is such that $C_0^{\mathcal{I}} = \{a_0\}$.

PROPERTIES OF THE TABLEAU ALGORITHM

Let's call our tableau algorithm `CSat` (for concept satisfiability).

To show that `CSat` is a decision procedure, we must show:

Termination: The algorithm `CSat` always terminates.

Soundness: `CSat` outputs "yes" on input $C_0 \Rightarrow C_0$ is satisfiable.

Completeness: C_0 satisfiable $\Rightarrow$ `CSat` will output "yes".

Subconcepts of a concept:

$$
\begin{aligned}
\mathsf{sub}(A) &= \{A\} \\
\mathsf{sub}(\neg C) &= \{\neg C\} \cup \mathsf{sub}(C) \\
\mathsf{sub}(\exists r.C) &= \{\exists r.C\} \cup \mathsf{sub}(C) \\
\mathsf{sub}(\forall r.C) &= \{\forall r.C\} \cup \mathsf{sub}(C) \\
\mathsf{sub}(C_1 \sqcup C_2) &= \{C_1 \sqcup C_2\} \cup \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2) \\
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Role depth of a concept:

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\mathrm{depth}(A) = \mathrm{depth}(\top) = \mathrm{depth}(\bot) &= 0 \\
\mathrm{depth}(\neg C) &= \mathrm{depth}(C) \\
\mathrm{depth}(\exists r.C) = \mathrm{depth}(\forall r.C) &= \mathrm{depth}(C) + 1 \\
\mathrm{depth}(C_1 \sqcup C_2) = \mathrm{depth}(C_1 \sqcap C_2) &= \max(\mathrm{depth}(C_1), \mathrm{depth}(C_2))
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3. if $D(b) \in \mathcal{A}$ and the unique path from a_0 to b has length k, then $\mathsf{depth}(D) \leq \mathsf{depth}(C_0) - k$

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Thus: bound on the size of ABoxes generated by the procedure

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Thus: bound on the size of ABoxes generated by the procedure

The tableau procedure only adds assertions to ABoxes
$\Rightarrow$ eventually all ABoxes will contain a clash or will be complete

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Suppose that `CSat` returns "yes" on input C_0.

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Claim: $\mathcal{I}$ is such that $C_0^{\mathcal{I}} \neq \emptyset$

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Claim: $\mathcal{I}$ is such that $C_0^{\mathcal{I}} \neq \emptyset$

To show the claim, we prove by induction on the size of concepts:

$$D(b) \in \mathcal{A} \quad \Rightarrow \quad b \in D^{\mathcal{I}}$$

SOUNDNESS OF CSAT (2)

Base case: $D = A$ or $D = \neg A$ or $D = \top$ or $D = \bot$

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If $D = A$, then $b \in A^{\mathcal{I}}$.
If $D = \neg A$, then $A(b) \notin \mathcal{A}$, so $b \in \neg A^{\mathcal{I}}$.
If $D = \top$, trivially $b \in \top^{\mathcal{I}} = \Delta^{\mathcal{I}}$. Cannot have $D = \bot$ since clash-free.

Induction hypothesis (IH): suppose holds whenever $|D| \leq k$

Induction step: show statement holds for D with $|D| = k + 1$

Again, many cases to consider:

- $D = E \sqcap F$: since $\mathcal{A}$ is complete, it must contain both $E(b)$ and $F(b)$. Applying the IH, we get $b \in E^{\mathcal{I}}$ and $b \in F^{\mathcal{I}}$, hence $b \in (E \sqcap F)^{\mathcal{I}}$
- $D = \exists r.E$: since $\mathcal{A}$ is complete, there exists c such that $r(b, c) \in \mathcal{A}$ and $E(c) \in \mathcal{A}$. Then $(b, c) \in r^{\mathcal{I}}$. From IH, get $c \in E^{\mathcal{I}}$, so $b \in (\exists r.E)^{\mathcal{I}}$
- $D = E \sqcup F$: left as practice
- $D = \forall R.E$: left as practice

Suppose that the concept C_0 is satisfiable.

Then the ABox $\{C_0(a_0)\}$ must be satisfiable too.

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We start with a satisfiable ABox and the rules are satisfiability-preserving, so eventually we will reach a complete, satisfiable (thus: clash-free) ABox and output 'yes'.

Bad news: our algorithm may require exponential time and space...

To see why, consider what happens if we run `CSat` on the concept

$$\bigsqcap_{0 \leq i < n} \underbrace{\forall r. \ldots \forall r.}_{i \text{ times}} (\exists r.B \sqcap \exists r.\neg B)$$

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Good news: can modify algorithm so it runs in polynomial space

- instead of set of ABoxes, keep only 1 ABox in memory at a time
  - when apply the $\sqcup$-rule, first examine $\mathcal{A}_1$, then afterwards examine $\mathcal{A}_2$
  - remember that second disjunct stills needs to be checked
- explore the children of an individual one at a time
  - possible because no interaction between the different "branches"
  - store which $\exists r.C$ concepts have been tested, which are left to do
- this allows us to keep at most $|C_0|$ individuals in memory at a time

Hierarchy of complexity classes

$$\text{PTIME} \subseteq \text{NP} \subseteq \ldots \subseteq \text{PSPACE} \subseteq \text{EXPTIME} \subseteq \text{NEXPTIME} \subseteq \ldots \subseteq \text{EXPSPACE} \ \ldots$$

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Theorem: $\mathcal{ALC}$ concept satisfiability (no TBox) is PSPACE-complete.

- Membership in PSPACE shown using modified tableau procedure
- Hardness for PSPACE shown by giving a reduction from some known PSPACE-hard problem (e.g. QBF validity)

EXTENSION TO KB SATISFIABILITY

Now we want to modify the algorithm to handle KB satisfability.

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Concretely, we might try adding the following rule:

> **TBox rule** if a is in $\mathcal{A}$, $C \sqsubseteq D \in \mathcal{T}$, & $(\mathtt{NNF}(\neg C) \sqcup \mathtt{NNF}(D))\,(a) \notin \mathcal{A}$
> then replace $\mathcal{A}$ with $\mathcal{A} \cup \{(\mathtt{NNF}(\neg C) \sqcup \mathtt{NNF}(D))(a)\}$

EXAMPLE: NON-TERMINATION

Let's try the modified procedure on the KB $(\{F \sqsubseteq \exists S.F\}, \{F(a)\})$

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Seems we have a problem... How can we ensure termination?

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Formally: given individuals a, b from $\mathcal{A}$, we say that b **blocks** a if:

- $\{C \mid C(a) \in \mathcal{A}\} \subseteq \{C \mid C(b) \in \mathcal{A}\}$
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Modify rules so that they **only apply to unblocked individuals**.

$\sqcap$-rule: if $(C_1 \sqcap C_2)(a) \in \mathcal{A}$, a is not blocked, and $\{C_1(a), C_2(a)\} \not\subseteq \mathcal{A}$, then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a), C_2(a)\}$

$\sqcup$-rule: if $(C_1 \sqcup C_2)(a) \in \mathcal{A}$, a is not blocked, and $\{C_1(a), C_2(a)\} \cap \mathcal{A} = \emptyset$, then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a))\}$ **and** $\mathcal{A} \cup \{C_2(a))\}$

$\forall$-rule: if $\{\forall r.C(a), r(a,b)\} \in \mathcal{A}$, a is not blocked, and $C(b) \notin \mathcal{A}$, then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C(b))\}$

$\exists$-rule: if $\{\exists r.C(a)\} \in \mathcal{A}$, a is not blocked, and no $\{r(a,b), C(b)\} \subseteq \mathcal{A}$, then **pick a new individual name** d and replace $\mathcal{A}$ with $\mathcal{A} \cup \{r(a,d), C(d)\}$

$\sqsubseteq$-rule: if a appears in $\mathcal{A}$ and a is not blocked, $C \sqsubseteq D \in \mathcal{T}$, and $(\mathtt{NNF}(\neg C) \sqcup \mathtt{NNF}(D))(a) \notin \mathcal{A}$, then replace $\mathcal{A}$ with $\mathcal{A} \cup \{(\mathtt{NNF}(\neg C) \sqcup \mathtt{NNF}(D))(a)\}$

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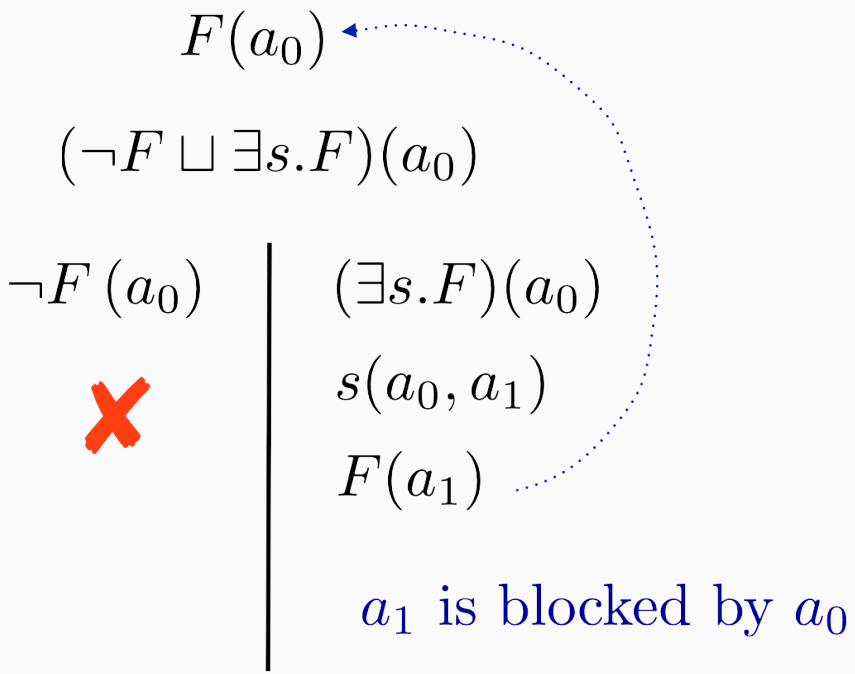

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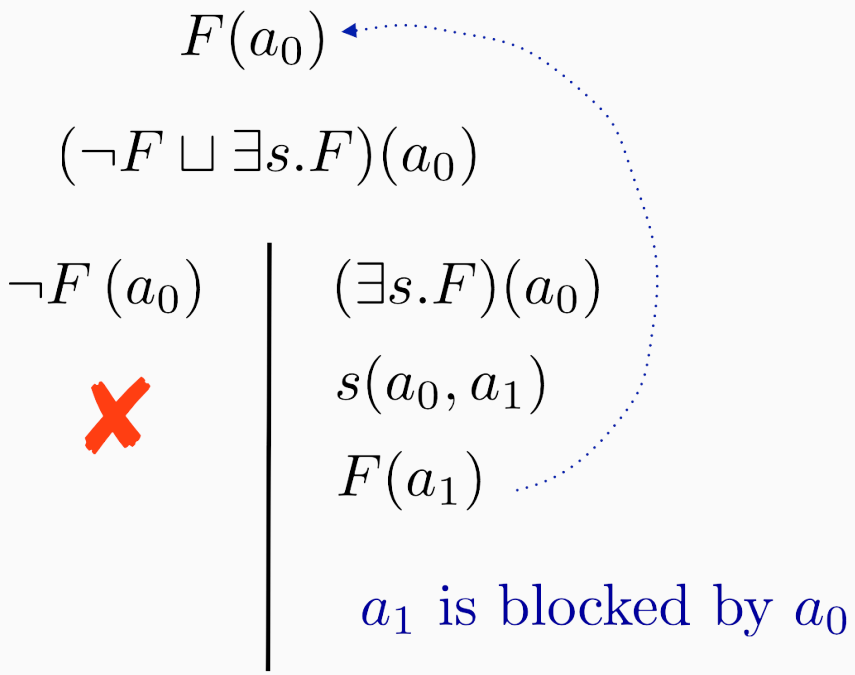

We obtain a complete and clash-free ABox $\Rightarrow$ the KB is satisfiable

ANOTHER BLOCKING EXAMPLE

Consider the TBox $\mathcal{T} = \{A \sqsubseteq \exists r.A, A \sqsubseteq B, \exists r.B \sqsubseteq D\}$ and suppose we want to test whether $\mathcal{T} \models A \sqsubseteq D$.

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Observe: individual can be blocked, then later become unblocked

PROPERTIES OF KBSAT

Let's call our new tableau algorithm `KBSat` (for KB satisfiability).

Termination: The algorithm `KBSat` always terminates.

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So: KBSat is a decision procedure for KB satisfiability.

COMPLEXITY OF REASONING IN EXPRESSIVE DLS

Tableau procedure takes **exponential time and space**

- can have exponentially long 'branches' to explore

Complexity results tell us this is **unavoidable in worst case**:

Theorem: In $\mathcal{ALC}$, KB satisfiability is **EXPTIME-complete**

- for highly expressive DLs ($\rightsquigarrow$ OWL 2): complexity even higher

OPTIMIZATIONS

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- examine source of conflicts to prune search space (backjumping)
- reduce number of $\sqcup$'s created by TBox inclusions (absorption)
- reduce number of satisfiability checks during classification

ABSORPTION (1)

When $\mathcal{T} = \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n\}$, we get n disjunctions per individual:

$$(\mathtt{NNF}(\neg C_1) \sqcup \mathtt{NNF}(D_1))(a), \ldots, (\mathtt{NNF}(\neg C_n) \sqcup \mathtt{NNF}(D_n))(a)$$

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Good news: we've lowered the number of disjunctions!

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Absorption technique:

1. preprocess the TBox by replacing inclusions with equivalent inclusions with atomic concept on left, whenever possible
2. when running tableau algorithm
   - use new $\sqsubseteq^{at}$-rule for inclusions $A \sqsubseteq D$ with A a concept name
   - use regular $\sqsubseteq$-rule for the other TBox inclusions

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Let's use absorption on the KB $(\mathcal{T}, \{A(a)\})$ with:

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Result: avoid disjunction, algorithm terminates much faster!

OPTIMIZATIONS FOR CLASSIFICATION

Classification: find all pairs of concept names A, B with $\mathcal{T} \models A \sqsubseteq B$

Naïve approach: test satisfiability of $A \sqcap \neg B$ w.r.t. $\mathcal{T}$ for all pairs A, B

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Some ideas:

- some cases are obvious
  - $A \sqsubseteq A$ and inclusions that are explicitly stated in $\mathcal{T}$
- use simple reasoning to obtain new (non-)entailments
  - if know $\mathcal{T} \models A \sqsubseteq B$ and $\mathcal{T} \models B \sqsubseteq D$, then $\mathcal{T} \models A \sqsubseteq D$
  - if know $\mathcal{T} \models A \sqsubseteq B$ and $\mathcal{T} \not\models A \sqsubseteq D$, then $\mathcal{T} \not\models B \sqsubseteq D$