Handout: Description logic $\mathcal{EL}$

Syntax of $\mathcal{EL}$

$\mathcal{EL}$ allows complex concepts of the following form:

\[ C := \top | A | C_1 \cap C_2 | \exists r.C \]

where $A$ is a concept and $r$ a role name.

An $\mathcal{EL}$ TBox contain concept inclusions $C_1 \sqsubseteq C_2$ (with $C_1, C_2$ as above)

We say that an $\mathcal{EL}$ TBox $\mathcal{T}$ is in normal form if it contains only inclusions of the following forms:

\[ A \sqsubseteq B \quad A_1 \cap A_2 \sqsubseteq B \quad A \sqsubseteq \exists r.B \quad \exists r.A \sqsubseteq B \]

where $A, A_1, A_2, B$ are concept names (or $\top$).

Saturation rules

\[ \frac{A \sqsubseteq A}{A \sqsubseteq A} \quad \frac{A \sqsubseteq \top}{A \sqsubseteq \top} \quad \frac{A \sqsubseteq B \quad B \sqsubseteq D}{A \sqsubseteq D} \quad \frac{A \sqsubseteq \exists r.B_1 \quad B_1 \sqsubseteq B_2 \quad \exists r.B_2 \sqsubseteq D}{A \sqsubseteq D} \]

Instantiated rule:

- obtained from one of the ‘abstract’ saturation rules above by replacing $A, B, D$ by $\mathcal{EL}$-concepts and $r$ by some role name

- important: after replacement, can only contain axioms that are in normal form and concept assertions using (non-complex) concept names

Premises of (instantiated) rule = axioms / assertions above the line

Conclusion of (instantiated) rule = axiom / assertion below the line
Saturation procedure

Assume w.l.o.g. that start from KB whose TBox is in normal form & whose ABox contains $\top(a)$ for each of its individuals $a$

Call an instantiated rule with applicable in $\mathcal{K}$ if its premises belong to $\mathcal{K}$ and its conclusion is not already in $\mathcal{K}$

Apply an instantiated rule (if applicable) = add its conclusion to the KB

Saturation procedure: exhaustively apply instantiated rules until no rule is applicable

Denote by $\text{sat}(\mathcal{K})$ or $\text{sat}(\mathcal{T}, \mathcal{A})$ (resp. $\text{sat}(\mathcal{T})$) the (unique) result of applying saturation procedure to KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ (resp. TBox $\mathcal{T}$)

Using saturated KB for reasoning

To find all instances of concept name $A$ w.r.t. $\mathcal{K} = (\mathcal{T}, \mathcal{A})$:
  1. Normalize $\mathcal{T}$, yielding $\mathcal{T}'$, then construct $\text{sat}(\mathcal{T}', \mathcal{A})$
  2. Return all individuals $c$ such that $A(c) \in \text{sat}(\mathcal{T}', \mathcal{A})$.

To test whether $\mathcal{T} \models A \sqsubseteq B$ ($A, B$ concept names):
  1. Normalize $\mathcal{T}$, yielding $\mathcal{T}'$, then construct $\text{sat}(\mathcal{T}')$ (or $\text{sat}(\mathcal{T}', \mathcal{A})$ if have an ABox $\mathcal{A}$)
  2. Check if $\text{sat}(\mathcal{T}')$ contains $A \sqsubseteq B$, return yes if so, else no.

For assertions / inclusions involving complex concepts:
  - Use new concept names to represent complex concepts, e.g. if $C$ is a complex concept, add $X_C \sqsubseteq C$ and $C \sqsubseteq X_C$ to $\mathcal{T}$ (with $X_C$ fresh).
  - Proceed as above but use $X_C$ in place of $C$.

Canonical model

Define canonical model $\mathcal{C}_\mathcal{K}$ of $\mathcal{K}$ as follows:

- $\Delta_{\mathcal{C}_\mathcal{K}} = \text{Ind}(\mathcal{A}) \cup \{w_A \mid A \text{ concept name appearing in } \mathcal{K}\} \cup \{w_\top\}$
- $A_{\mathcal{C}_\mathcal{K}} = \{b \mid A(b) \in \text{sat}(\mathcal{K})\} \cup \{w_B \mid B \sqsubseteq A \in \text{sat}(\mathcal{K})\}$
- $r_{\mathcal{C}_\mathcal{K}} = \{(a, b) \mid r(a, b) \in \mathcal{K}\} \cup \{(w_A, w_B) \mid A \sqsubseteq \exists r.B \in \text{sat}(\mathcal{K})\}$
  \[\cup \{(a, w_B) \mid A \sqsubseteq \exists r.B \in \text{sat}(\mathcal{K}), A(a) \in \text{sat}(\mathcal{K}) \text{ for some } A\}\]

where $\text{Ind}(\mathcal{A})$ is set of individual names in $\mathcal{A}$