Handout: Description logic $\mathcal{EL}$

Syntax of $\mathcal{EL}$

$\mathcal{EL}$ allows complex concepts of the following form:

$$C := \top | A | C_1 \cap C_2 | \exists r.C$$

where $A$ is a concept and $r$ a role name.

An $\mathcal{EL}$ TBox contain concept inclusions $C_1 \sqsubseteq C_2$ (with $C_1, C_2$ as above)

We say that an $\mathcal{EL}$ TBox $\mathcal{T}$ is in normal form if it contains only inclusions of the following forms:

$$A \sqsubseteq B \quad A_1 \sqcap A_2 \sqsubseteq B \quad A \sqsubseteq \exists r.B \quad \exists r.A \sqsubseteq B$$

where $A, A_1, A_2, B$ are concept names (or $\top$).

Saturation rules

$$\frac{}{A \sqsubseteq A} \text{ T1} \quad \frac{}{A \sqsubseteq \top} \text{ T2} \quad \frac{A \sqsubseteq B \quad B \sqsubseteq D}{A \sqsubseteq D} \text{ T3}$$

$$\frac{A \sqsubseteq B_1 \quad A \sqsubseteq B_2 \quad B_1 \sqcap B_2 \sqsubseteq D}{A \sqsubseteq D} \text{ T4} \quad \frac{A \sqsubseteq \exists r.B_1 \quad B_1 \sqsubseteq B_2 \quad \exists r.B_2 \sqsubseteq D}{A \sqsubseteq D} \text{ T5}$$

$$\frac{A \sqsubseteq B \quad A(c)}{B(c)} \text{ A1} \quad \frac{A_1 \sqcap A_2 \sqsubseteq B \quad A_1(c) \quad A_2(c)}{B(c)} \text{ A2} \quad \frac{\exists r.A \sqsubseteq B \quad r(c, d) \quad A(d)}{B(c)} \text{ A3}$$

Instantiated rule:

- obtained from one of the ‘abstract’ saturation rules above by replacing $A, B, D$ by $\mathcal{EL}$-concepts and $r$ by some role name

- important: after replacement, can only contain axioms that are in normal form and concept assertions using (non-complex) concept names

Premises of (instantiated) rule = axioms / assertions above the line

Conclusion of (instantiated) rule = axiom / assertion below the line
**Saturation procedure**

Assume w.l.o.g. that start from KB whose TBox is in normal form & whose ABox contains \( \top(a) \) for each of its individuals \( a \)

Call an instantiated rule with *applicable in* \( \mathcal{K} \) if its premises belong to \( \mathcal{K} \) and its conclusion is not already in \( \mathcal{K} \)

Apply an instantiated rule (if applicable) = add its conclusion to the KB

Saturation procedure: *exhaustively apply* instantiated rules until no rule is applicable

Denote by \( \text{sat}(\mathcal{K}) \) or \( \text{sat}(\mathcal{T}, \mathcal{A}) \) (resp. \( \text{sat}(\mathcal{T}) \)) the (unique) result of applying saturation procedure to KB \( \mathcal{K} = (\mathcal{T}, \mathcal{A}) \) (resp. TBox \( \mathcal{T} \))

**Using saturated KB for reasoning**

To *find all instances of concept name* \( A \) w.r.t. \( \mathcal{K} = (\mathcal{T}, \mathcal{A}) \):

1. Normalize \( \mathcal{T} \), yielding \( \mathcal{T}' \), then construct \( \text{sat}(\mathcal{T}', \mathcal{A}) \)
2. Return all individuals \( c \) such that \( A(c) \in \text{sat}(\mathcal{T}', \mathcal{A}) \).

To *test whether* \( \mathcal{T} \models A \sqsubseteq B \) (\( A, B \) concept names):

1. Normalize \( \mathcal{T} \), yielding \( \mathcal{T}' \), then construct \( \text{sat}(\mathcal{T}') \) (or \( \text{sat}(\mathcal{T}', \mathcal{A}) \) if have an ABox \( \mathcal{A} \))
2. Check if \( \text{sat}(\mathcal{T}') \) contains \( A \sqsubseteq B \), return yes if so, else no.

For *assertions / inclusions involving complex concepts*:

- Use new concept names to represent complex concepts, e.g. if \( C \) is a complex concept, add \( X_C \sqsubseteq C \) and \( C \sqsubseteq X_C \) to \( \mathcal{T} \) (with \( X_C \) fresh).
- Proceed as above but *use* \( X_C \) *in place of* \( C \).

**Canonical model**

Define canonical model \( \mathcal{C}_\mathcal{K} \) of \( \mathcal{K} \) as follows:

- \( \Delta^{\mathcal{C}_\mathcal{K}} = \text{Ind}(\mathcal{A}) \cup \{ w_A \mid A \text{ concept name appearing in } \mathcal{K} \} \cup \{ w_\top \} \)
- \( A^{\mathcal{C}_\mathcal{K}} = \{ b \mid A(b) \in \text{sat}(\mathcal{K}) \} \cup \{ w_B \mid B \sqsubseteq A \in \text{sat}(\mathcal{K}) \} \)
- \( r^{\mathcal{C}_\mathcal{K}} = \{ (a, b) \mid r(a, b) \in \mathcal{K} \} \cup \{ (w_A, w_B) \mid A \sqsubseteq \exists r. B \in \text{sat}(\mathcal{K}) \}
\cup \{ (a, w_B) \mid A \sqsubseteq \exists r. B \in \text{sat}(\mathcal{K}), A(a) \in \text{sat}(\mathcal{K}) \text{ for some } A \} \)

where \( \text{Ind}(\mathcal{A}) \) is set of individual names in \( \mathcal{A} \)