

Random pseudofractal networks with competition

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Received 21 September 2006; received in revised form 9 February 2007

Available online 19 March 2007

Abstract

In this paper, we present a simple rule which assigns fitness to each edge to generate random pseudofractal networks (RPNs). This RPN model is both scale-free and small-world. We obtain the theoretical results that the power-law exponent is $\gamma = 2 + 1/(1 + \alpha)$ for the tunable parameter $\alpha > -1$, and that the degree distribution is of an exponential form for others. Analytical results also show that an RPN has a large clustering coefficient and can process hierarchical structure as $C(k) \sim k^{-1}$ that is in accordance with many real networks. And we prove that the mean distance $L(N)$ scales slower logarithmically with network size N . In particular, we explain the effect of nodes with degree 2 on the clustering coefficient. These results agree with numerical simulations very well.

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Keywords: Growing networks; Scale-free feature; Small-world effect

1. Introduction

Many social, biological, computer, and physical systems can be described as complex networks with numerous nodes and edges between them [1,2]. Despite the differences in the meaning of the nodes and edges, complex networks display considerable statistical and topological similarities. The small-world effect [3] and the scale-free feature [4] are two significant discoveries of diverse complex networks.

The small-world networks (SWN) are usually characterized by two properties: a large clustering coefficient, which is a characteristic of regular networks, and a small mean distance, which is typically observed in random networks. Watts and Strogatz first introduced an SWN model (WS model) [3], which was constructed by starting with one-dimensional lattice and randomly rewiring each edge with a certain probability. However, there is a probability for the WS model to be broken into unconnected clusters. Subsequently, many other models have been presented and improved [5–10]. In random networks [11,12] as well as in the SWNs, the degree distribution $P(k)$, which is defined as the probability that a randomly selected node has k edges, decays exponentially. However, the degree distribution of many real networks obeys a power law: $P(k) \sim k^{-\gamma}$, where the power-law exponent γ usually ranges between 2 and 3. A growing scale-free network model (SFN) was first reported by Barabási and Albert (BA model) [4]. The BA model suggests that growth and preferential

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attachment are two basic ingredients of self-organization of SFN. In other words, networks grow continuously by adding new nodes. These added nodes are preferentially attached to the existing nodes with large degrees. With this idea, a large number of SFN models have been produced [13–16].

From the perspective of network properties, SW effect and SF feature are not exclusive in a great deal of networks [17–20]. Yet the above models have difficulty in capturing the two properties simultaneously. In recent years, a few researchers have resorted to mathematical methods to construct some deterministic network models, such as hierarchical organization [21], pseudofractal networks [22,23], and Apollonian networks [24]. These models are entirely deterministic networks without considering statistical mechanism. Then randomness is introduced to improve these models [25–28].

From the perspective of the detailed growth, preferential attachment is not always a natural hypothesis [29,30] since the information of a large involved network will not be available. Thus varying fitness models [31–33] have been proposed. In real-world networks, fitness [34], describing the ability to compete for edges, is related to some intrinsic quality of nodes, such as the wealth, the beauty, or the social skills of an individual. In general, a probability distribution $\rho(x)$ is assigned to each node of the network for a real non-negative variable x . Though it is easily applied to analyze in theory, the assumption needs prior knowledge of the probability distribution. Here we introduce a novel definition of fitness by a simple rule instead of a connection function. This rule can be simply explained by a food web in nature. In this web, we do not need to learn $\rho(x)$ for each animal x but a basic rule: the law of the jungle (of course, there are many other rules in food webs, here the law of the jungle is considered to be the most important one). According to the law, fitness of an animal can be defined as the “power”. This definition is quite easy and acceptable. The more powerful animals seize the top position in the food chain, in other words, they have a larger fitness. Every animal is competing for the chance of survival in a competitive environment.

The rest of this paper is organized as follows: In Section 2, we provide a prototype of a growing network model. Since they can be considered as a variation of pseudofractal networks, the networks are called random pseudofractal networks (RPNs) [28]. Section 3 gives analytic results of RPNs, including the SF feature and SW effect. In Section 4, numerical simulations and some discussions are reported. Our work is summarized in Section 5.

2. Model description

In RPNs, growth and competing attachment are two basic factors. Adding new nodes and edges continuously realizes the growth of networks, and these added nodes are connected to both ends of selected edges by a rule. Competing attachment is realized by assigning fitness according to the rule. As a simple demonstration, suppose that the fitness of a latter edge joining a cluster is half of the previous one (actually, we can adjust this value by a tunable parameter α in the following), and the nodes, coming into the same cluster simultaneously, are of the same fitness. Obviously, we need a concrete value to measure the fitness. As an attempt, we select the arc length of circumcircle in this paper.

The detailed mechanism of an RPN can be described as (Fig. 1): an RPN starts from an edge with two nodes. At each time step, a node with two edges is added to a selected edge present in the system with a certain probability. The probability of selecting a particular edge is determined by its fitness. In this network, we consider the fitness to be related to the arc length of the corresponding edge. An RPN can be generated with this rule. In this network, the total number of edges is $2t + 1$ at time t , and the mean degree of the whole network is 4 when t is large enough.

In order to explain the RPN clearly, we introduce some useful notations. Let $\{N_i | i = -1, 0, 1, 2, \dots\}$ denote nodes set, and $\overline{N_i N_j}$ denote an edge between nodes N_i and N_j . A triangle $\Delta N_{-1} N_0 N_1$ is composed of initial nodes N_{-1}, N_0, N_1 . All arcs are defined on the circumcircle of the triangle $\Delta N_{-1} N_0 N_1$. Then we denote by $\widehat{N_i N_j}$ an arc of the corresponding edge $\overline{N_i N_j}$. In particular, the arc $\widehat{N_i N_j}$ is the one whose length is less than half of the circle. Thus we can express their relationship by mathematical functions. At any time i , we label the new node by the time of its birth, N_i with $i \leq t$. Let $\hat{l}(0)$ be the length of initial arc $\widehat{N_{-1} N_0}$. According to the rule, a new node N_i is added to the network at the i th step. If the edge $\overline{N_j N_k}$ is selected at i th step, let $\hat{l}(i)$ and $\hat{l}(t + i)$ denote the length of new arcs, $\widehat{N_j N_i}$, $\widehat{N_k N_i}$, respectively. And the fitness of corresponding arcs are

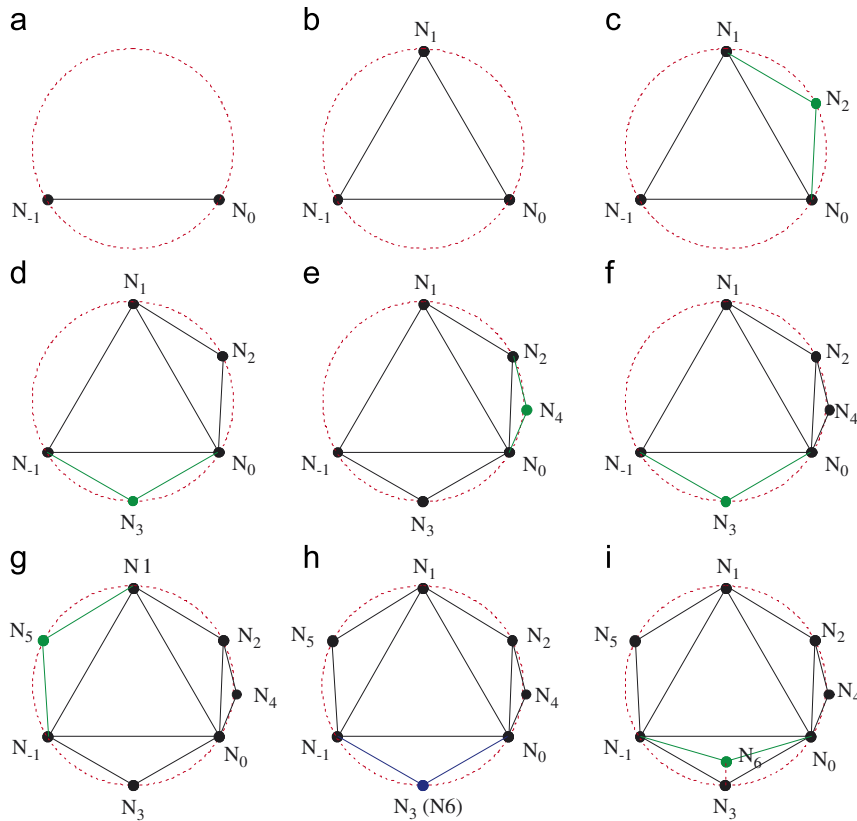


Fig. 1. Scheme of an RPN growth. The figures from (a) to (h) show a possible process for the RPN. The growth starts from an edge $\overline{N_{-1}N_0}$ with two nodes N_{-1}, N_0 at $t = 0$ (a). At the first time step $t = 1$, the edge $\overline{N_{-1}N_0}$ is selected (actually, we can also say that the arc $\widehat{N_{-1}N_0}$ is selected), and a new node N_1 , and two edges $\overline{N_{-1}N_1}, \overline{N_0N_1}$ are all added to the growing network (b). Then we obtain a triangle $\triangle N_{-1}N_0N_1$ and its circumcircle (red dashed line). At $t = 2$, the edge $\overline{N_0N_1}$ is selected with a certain probability, then node N_2 and two edges $\overline{N_1N_2}, \overline{N_0N_2}$ are added to the midpoint of the arc $\widehat{N_0N_1}$ (c). With this rule, one can easily get an RPN. Obviously, a midpoint may be selected more than once shown in (h). In fact, N_3 and N_6 are different ones (i).

labelled as $f(i)$ and $f(t + i)$. For brevity, let $\hat{l}(0) = \hat{l}(1) = \hat{l}(t + 1) = 1$, and the newly added node lie on the midpoint of the arc, namely

$$\hat{l}(i) = \hat{l}(t + i) = \frac{1}{2}\hat{l}(i - 1). \tag{1}$$

Thus, $\triangle N_{-1}N_0N_1$ is an equilateral one. All nodes can be on the circumcircle of this triangle. Moreover, define

$$f(i) = \hat{l}^\alpha(i), \tag{2}$$

where α is competing exponent. Compared with the latter ones, $\alpha > 0$ represents the former edges being relatively strong competing ability and vice versa.

Note that the model will reduce to a special RPN [28] with a power-law exponent $\gamma = 3$ when $\alpha = 0$. In particular, the mechanism of this special network can be explained by growth and preferential attachment [28]. As a result, the special RPN looks like a BA model. However, the location of nodes is the key difference between the RPN and the BA model. In RPNs, all nodes can lie in a ring (in this paper, we choose the circumcircle of triangle determined by initial three nodes) constructing a large cluster, which leads to an SW effect.

3. Analytical solutions

Generally speaking, three mathematical quantities including degree distribution, clustering coefficient, and mean distance are standard properties of various complex networks. This section will focus on these three parameters.

Mean arc length. Let mean arc length $\bar{l}(i)$ be the expectation value of $\hat{l}(i)$. Then at time t , the probability that a node N_j is selected is

$$P(j, t) = \frac{\bar{l}^\alpha(j)}{S(t)}, \tag{3}$$

where $S(t) = 1 + 2\sum_{j=1}^t \bar{l}^\alpha(j)$, is the fitness sum at time t . Furthermore, the evolution equation can be written as

$$\bar{l}(t) = \sum_{i=1}^{t-1} \bar{l}(i)P(i, t-1). \tag{4}$$

When t is large enough and $S(t)$ satisfies the condition $S(t) \gg 1$, $\bar{l}(t)$ can be rewritten in a continuous form

$$\bar{l}(t) = \frac{\int_0^t \bar{l}^{1+\alpha}(u) du}{2 \int_0^t \bar{l}^\alpha(u) du}, \tag{5}$$

one solution is $\bar{l}(t) = c_0 t^{-\beta}$, where c_0 is a constant.

Substituting the above result into Eq. (5), we get

$$\beta = \frac{1}{\alpha + 2}. \tag{6}$$

Note that we need to check the inequality $\bar{S}(t) \gg 1$, and easily derive the condition $\alpha \geq -1$. Without special statements, we always assume that β ranges between 0 and 1 in the following.

Mean node strength. Denote the mean strength [35,36] of node N_i as $\bar{s}(i, t)$ at time t . In an RPN, the mathematical expression can be defined by

$$\bar{s}(i, t) = 2\bar{f}(i) + \sum_{j>i, j \in V(i)} \bar{f}(j) \tag{7}$$

at t time step, where $\bar{f}(i) = \bar{l}^\alpha(i)$, and $V(i)$ is the neighbor set of node N_i . The first term on the right-hand side of Eq. (7) is determined by the birth of N_i , while the second term represents the changes because of other edges after N_i . According to the evolution of RPNs, we can obtain

$$\bar{s}(i, t) = \bar{s}(i, t-1) + \frac{\bar{s}(i, t-1)\bar{f}(t)}{1 + 2\sum_{j=1}^{t-1} \bar{f}(j)}. \tag{8}$$

Using the expression $\bar{s}(i, t) - \bar{s}(i, t-1) = \partial \bar{s}(i, t) / \partial t$, we can derive the continuous form

$$\frac{\partial \bar{s}(i, t)}{\partial t} = \frac{\bar{s}(i, t)\bar{f}(t)}{2 \int_0^t \bar{f}(u) du}, \tag{9}$$

accounting for the boundary condition $\bar{s}(i, i) = 2f(i)$, we obtain the solution,

$$\bar{s}(i, t) = 2c_0^\alpha t^\beta i^{\beta-1}. \tag{10}$$

Degree distribution. The degree distribution of the node N_i in the continuous approximation is $p(k, i, t) = \delta(k - \bar{k}(i, t))$ [14], where $\delta(\cdot)$ is the δ -function and $\bar{k}(i, t)$ is the mean degree of the node N_i at time t . Then,

$$\frac{\partial \bar{k}(i, t)}{\partial t} = \frac{\bar{s}(i, t)}{2 \int_0^t \bar{l}^\alpha(u) du}, \tag{11}$$

where $\bar{k}(t, t) = 2$. Applying $\int_i^t dt$ to both sides of Eq. (11), we obtain the solution,

$$\bar{k}(i, t) = \frac{2\beta}{1-\beta} \left(\frac{i}{t}\right)^{\beta-1} + 2 - \frac{2\beta}{1-\beta}. \tag{12}$$

In the interval $0 < \beta < 1$ and $1 < i \ll t$, we get $\bar{k}(i, t) \sim (i/t)^{-(1-\beta)}$. In the continuous approximation [22], one may find the distribution $P(k, t)$ using the obtained $\bar{k}(i, t)$,

$$P(k, t) = -\frac{1}{t} \left(\frac{\partial \bar{k}(i, t)}{\partial i}\right)^{-1}. \tag{13}$$

Therefore, the degree distribution can be written as

$$P(k, t) = \frac{1}{2\beta} \left(\frac{1-\beta}{2\beta}\right)^{-\gamma} \left(k + \frac{2\beta}{1-\beta} - 2\right)^{-\gamma}, \tag{14}$$

where $\alpha > -1$ and $\gamma = 2 + 1/(1 + \alpha)$. When $k \gg 2 - 2\beta/(1 - \beta)$, the degree distribution obeys a power law: $P(k, t) \sim k^{-\gamma}$. Actually the term $\beta/(1 - \beta)$ will be so large that it cannot be ignored when β is close to 1. In this case, the degree distribution is not a power law but exponential such that

$$P(k, t) \sim \exp\left[-k\left(\frac{2-\beta}{2\beta} - 1\right)\right] \tag{15}$$

In particular, we consider a special case. When $\alpha = -1$, $\bar{k}(i, t) = 2t/c_0$, and Eq. (11) degenerates into $\partial \bar{k}(i, t)/\partial t = 2/t$. Then we can easily derive the solution $\bar{k}(i, t) = 2 \ln(t/i) + 2$, and the degree distribution satisfies $P(k) \sim \exp(-k/2)$, following an exponential form.

Clustering coefficient. The clustering coefficient C measures the extent of the interconnection of node’s neighbors. In general, the clustering coefficient $c(i)$ of a node N_i has the following form:

$$c(i) = \frac{2V(i)}{k(i)(k(i) - 1)}, \tag{16}$$

where $k(i)$ is the degree of the node N_i . The clustering coefficient of the whole network is the mean of all individuals. When N_i is added to an RPN, the degree of this node is 2 and $V(i) = 1$. If the degree of N_i increases by 1, $V(i)$ will increase by 1 at the same time. Namely, $V(i) = k(i) - 1$. Then the clustering function $C(k) \sim k^{-1}$ is consistent with most empirical data. The expression in a continuous form follows,

$$C = \int_{k_{min}}^{k_{max}} \frac{P(k)}{k} dk, \tag{17}$$

where k_{min} and k_{max} are the minimal and maximal degrees, respectively. It is clear $k_{min} = 2$ and for sufficiently large t , $k_{max} \gg k_{min}$. Then we give the approximate expression by theoretical analysis

$$C(\alpha) = \frac{\alpha + 2}{2\alpha + 3}, \quad \alpha \geq -1. \tag{18}$$

In above computational process, $P(k) = (\gamma - 1)2^{\gamma-1}k^{-\gamma}$ when the degree distribution of the RPN shows a power law, where the coefficient of the degree distribution is determined by

$$\int_{k_{min}}^{k_{max}} P(k) dk = 1. \tag{19}$$

However, the result in Eq. (18) does not agree with numerical simulation because of the imprecision of degree distribution. We will have a further discussion in Section 4.

Mean distance. Mean distance is another important parameter that indicates whether a network is of an SW effect or not. Consider an undirected network with t nodes, we define $L(t)$ to be the mean distance between node pairs in a network:

$$L(t) = \frac{2\sigma(t)}{t(t+1)}, \tag{20}$$

where $\sigma(t) = \sum_{i \leq j} d_{ij}$ is the total distance of a network with t nodes, and d_{ij} denotes the distance between N_i and N_j . Strictly speaking, a network is said to show the SW effect if the value of $L(t)$ scales logarithmically or slower with network size t for fixed mean degree. The mean distance $L(t)$ in an RPN is satisfied with this relation:

$$\sigma(t+1) < \sigma(t) + t + \frac{2\sigma(t)}{t}. \quad (21)$$

The proof is omitted here since it is similar to Ref. [26]. The solution is $\sigma(t) = t^2(\ln t + c_1)$ when Eq. (21) is written as an equation, where c_1 is a constant. Thus, we derive that $L(t)$ grows a little slower than $\ln t$.

4. Numerical simulation and discussion

In this section, we simulate the RPN with $|\alpha| < 2$ to verify our analytical results. Fig. 2 shows the simulation of a power-law distribution, which agrees very well with the analytic one. When the competing parameter α is close to -1 , the form of degree distribution changes from a power law to exponential. Fig. 3 shows that the degree distribution is exponential, which agrees well with the analysis. Then a question appears: What about $\alpha < -1$? It is shown that the degree distribution in RPNs also decays exponential in the interval $(-2, -1)$ by simulation (Fig. 4). However, the distribution $P(2)$, where $P(2) \gg P(k)$ with $k > 2$, does not match the line in $k - \log P(k)$ coordinates. In particular, $\bar{k}(i, t)$ is almost constant when α is smaller than -2 . That is to say, the degrees of most nodes in such network are two. Then the network is similar to the NW model since the lattice is a ring and those edges (in particular the early ones) whose degrees are more than two can be regarded as “shortcuts” in SW networks.

From the above discussion and simulation, the degree distribution of RPNs will evolve from exponential to a power law as α gradually increases. The maximal degree k_{max} of this network will increase with α . In particular, the power-law exponent γ in this network ranges in $2 < \gamma \leq 3$ when α is equal to or bigger than zero.

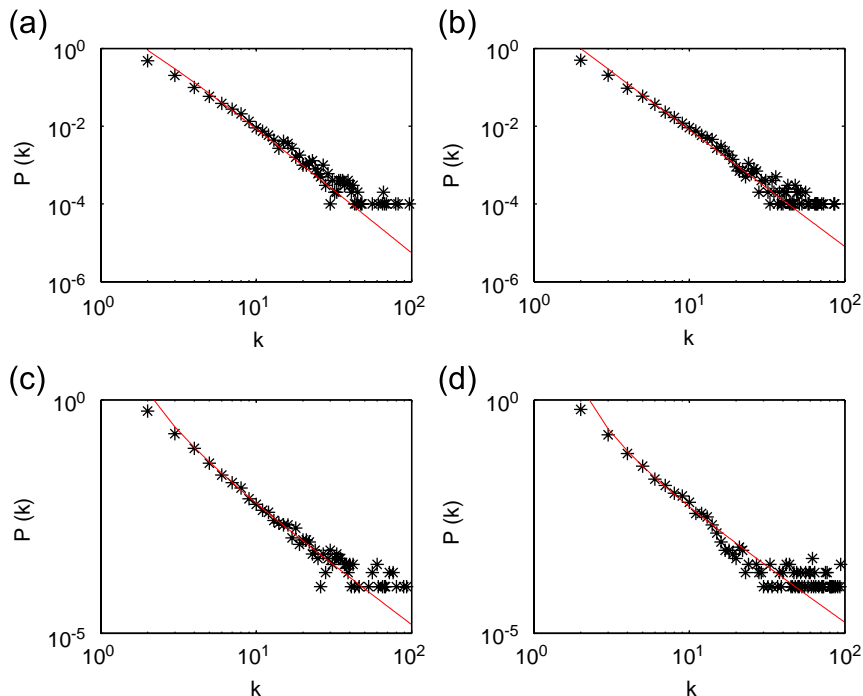


Fig. 2. The power-law degree distribution with the network size $N = 10^5$. The figures from (a) to (d) show the degree distribution of an RPN with $\alpha = -0.2$ (a), $\alpha = 0$ (b), $\alpha = 1$ (c), and $\alpha = 1.9$ (d). In these figures, the curves (red) are the theoretical results, where the power-law exponents are $\gamma = 2 + 1/(\alpha + 1)$, and the stars (black) are the simulation results.

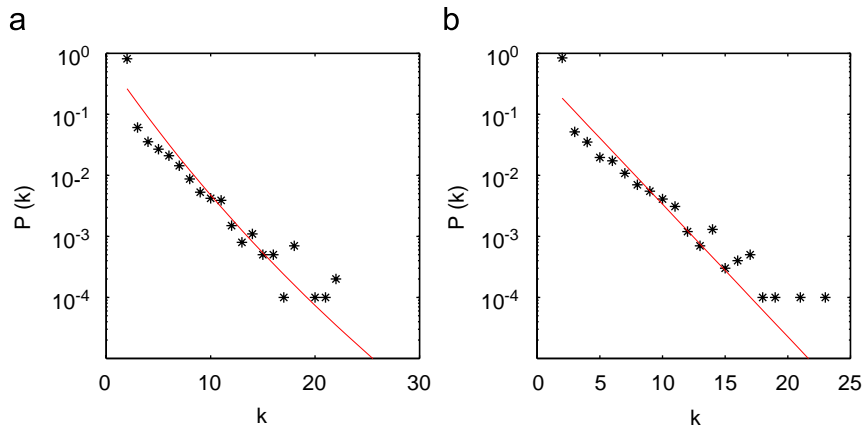


Fig. 3. The exponential degree distribution with the network size $N = 10^5$. The figures show the degree distribution of an RPN, with $\alpha = -0.95$ (a), $\alpha = -1$ (b). In these figures, the curves (red) are the theoretical results, while the stars (black) are the simulation results. The slopes in the two degree distributions are $\kappa = -0.55, -0.5$, respectively.

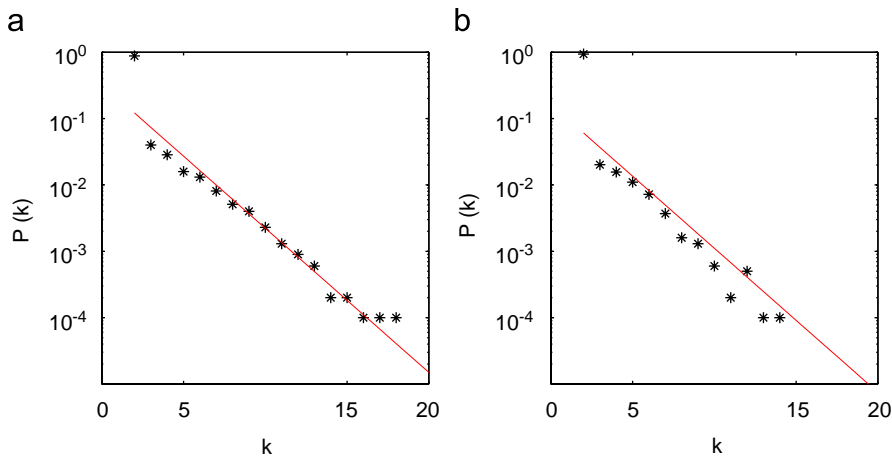


Fig. 4. The exponential degree distribution with the network size $N = 10^5$. The figures show the degree distribution of an RPN, with $\alpha = -1.2$ (a), and $\alpha = -1.8$ (b). In the two figures, the curves (red) are the theoretical results, while the stars (black) are the simulation results. The slopes in the two figures are the same, $\kappa = -1/2$, which is independent of α . This result cannot be derived by theoretical analysis since the condition $\alpha > -1$ holds.

According to Eq. (18), the clustering coefficient will monotonically decrease when α is bigger than 0. The numerical simulation in Fig. 5 shows the opposite. One key reason is the application of an imprecise degree distribution to clustering coefficient. Take $\alpha = 0$ for example. The exact solution of degree distribution [22,28] is

$$P(k) = \frac{12}{k(k+1)(k+2)}, \tag{22}$$

which is a little different from the approximate solution $P(k) = 8k^{-3}$. According to the two expressions of degree distribution, we derive the clustering coefficients are 0.74 and 0.66, respectively. Obviously, the exact distribution of finite degree directly determines the clustering coefficient when k_{max} is big enough. On one hand, the clustering coefficient can be written as:

$$C(\alpha) = \frac{1}{N} \left(\sum_1^{n_2} C_2(\alpha) + \sum_{i=3}^{k_{max}} \sum_1^{n_i} C_i(\alpha) \right), \tag{23}$$

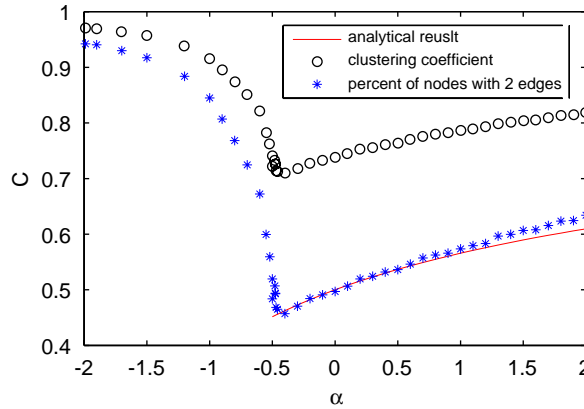


Fig. 5. The clustering function (○) of RPNs, and percentage of degree 2 of nodes (*), where the network size $N = 10^5$. The curve (red) is in the form: $p(x) = n_2/N = 1 - \alpha\beta$ in the interval $(-0.5, 2)$, which agrees well with the simulation results.

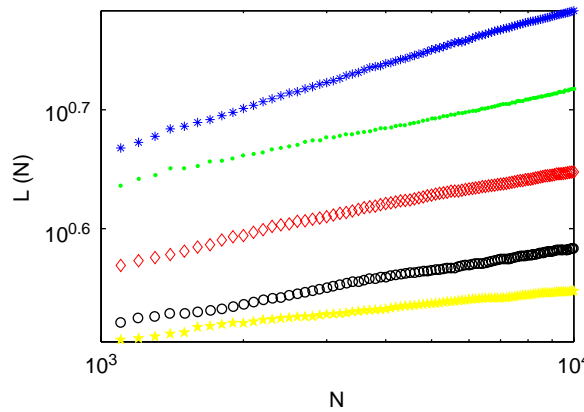


Fig. 6. The dependence between mean distance and the network size N of RPNs with $\alpha = 0$ (*), $\alpha = 0.2$ (·), $\alpha = 0.5$ (◇), $\alpha = 1.0$ (○), and $\alpha = 1.5$ (★). One can see that L increases very slowly as N increases. Here we just give the numerical results with $\alpha > 0$ since we focus on the RPNs with SF features.

where C_i is the clustering coefficient of a node with i edges, n_i is the number of nodes with i edges. In particular, $C_2(\cdot) \equiv 1$. From Fig. 5, the curve of $C(\alpha)$ is similar to that of the percentage of nodes with degree 2. Then n_2 is a very important parameter to deduce the expression of clustering coefficient. On the other hand, we assume $P(k) = ck^{-\gamma}$ in Eq. (18), but this function is not suitable for n_2 in simulation (Fig. 4). So we need to recalculate this parameter. Let $v(t)$ be the total fitness of nodes with two edges, then

$$\frac{dv(t)}{dt} = 2 \left(1 - \frac{v(t)}{2 \int_0^t \bar{f}(u) du} \right) \bar{f}(t) + \frac{2v(t)}{2 \int_0^t \bar{f}(u) du} \left(\bar{f}(t) - \frac{\bar{v}(t)}{t} \right). \tag{24}$$

One solution is $v(t) = ac_0^\alpha t^{-2\beta}$, where $a = \sqrt{\alpha + 9/4} - 1/2$. And we easily obtain

$$\frac{dn_2}{dt} = 1 - \frac{v(t)}{2 \int_0^t \bar{f}(u) du}. \tag{25}$$

Then $n_2 = (1 - \alpha\beta)t$, which agrees very well with the simulation in Fig. 5. The asymptotic behavior of the clustering coefficient is

$$C(\alpha) = 1 - \frac{a(3\alpha + 5)}{3(\alpha + 2)(2\alpha + 3)} \quad \text{if } \alpha > -0.5. \tag{26}$$

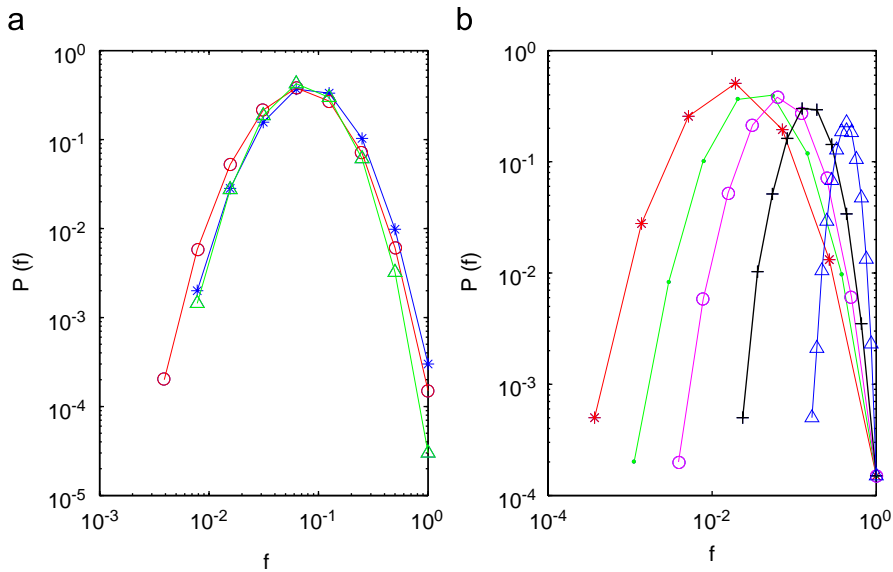


Fig. 7. (a) Shows the simulation of the fitness distribution with $N = 5 \times 10^3$ (*), $N = 10^4$ (o), and $N = 5 \times 10^4$ (Δ), where $\alpha = 1$. (b) Shows the simulation of fitness distribution of various α with $N = 10^4$, where $\alpha = 1.9$ (Δ), $\alpha = 1.4$ (+), $\alpha = 1.0$ (o), $\alpha = 0.6$ (\cdot), and $\alpha = 0.2$ (*).

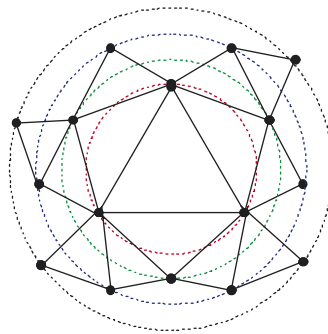


Fig. 8. Scheme illustration of the hierarchical structure of the growing network. The circles from the inside to outside are labelled as O_1, O_2, O_3, O_4 in order. There are only three nodes on circle O_1 , which are the initial ones. Then a new node will be on the circle O_{i+1} if the other two nodes constructing a new triangle with the new node are on the circle O_i, O_j , respectively, where $i > j$. With this rule, the newly added nodes will lie on the other outside circles. In particular, the nodes on the same circle will not be connected by one edge except the initial ones.

The clustering coefficient C is close to 1 when α is around -2 . One key reason is that the degree of most nodes is 2 in RPNs. Fig. 6 shows that the mean distance that satisfies $L(N) \sim \ln N$. The two figures indicate that RPN exhibits an SW effect. In Fig. 7, we give a simulation of the fitness distribution $P(f, t)$ in log–log scale. Notably, the fitness distribution does not obey a power law.

According to the above analysis, an RPN shows the SF property (of course, this property does not hold true for all α) and the SW effect in a simple growing way. In fact, there exists a hierarchical structure in this network with respect to fitness (shown in Fig. 8).

5. Summary

In this paper, we have presented an RPN by assigning fitness to newly added edges. The rule of fitness is quite simple and leads to some interesting properties of networks. We investigated some properties in detail

including degree distribution, clustering coefficient, and mean distance. This model has an SF (or an exponential) behavior that depends on the competing parameter α , and an SW effect with a large clustering coefficient and a small mean distance. Unfortunately, the fitness distribution is difficult to obtain, and the degree distribution of an RPN is not accurate. For example, the function $P(k, \alpha)$ is not suitable for calculating the clustering coefficient. And the case $\alpha < -1$ cannot be derived by this mathematical method. These are all left for our future work.

Acknowledgment

The authors thank the referees and the editor for their valuable comments and suggestions. This work is supported by the Chinese National Natural Science Foundation under Grant 60304018, the Technology Programs of Education Department of Anhui Province 2005jq1033.

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