The neighbour-sum-distinguishing edge-colouring game

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Abstract

Let $\gamma : E(G) \longrightarrow \mathbb{N}^*$ be an edge colouring of a graph G and $\sigma_\gamma : V(G) \longrightarrow \mathbb{N}^*$ the vertex colouring given by $\sigma_\gamma(v) = \sum_{e \ni v} \gamma(e)$ for every $v \in V(G)$. A *neighbour-sum-distinguishing edge-colouring* of G is an edge colouring γ such that for every edge uv in G, $\sigma_\gamma(u) \neq \sigma_\gamma(v)$. The *neighbour-sum-distinguishing edge-colouring game* on G is the 2-player game defined as follows. The two players, Alice and Bob, alternately colour an uncoloured edge of G. Alice wins the game if, when all edges are coloured, the so-obtained edge colouring is a neighbour-sum-distinguishing edge-colouring of G. Otherwise, Bob wins.

In this paper we study the neighbour-sum-distinguishing edge-colouring game on various classes of graphs. In particular, we prove that Bob wins the game on the complete graph K_n, $n \geq 3$, whoever starts the game, except when $n = 4$. In that case, Bob wins the game on K_4 if and only if Alice starts the game.

Keywords: neighbour-sum-distinguishing edge-colouring, neighbour-sum-distinguishing edge-colouring game, edge-weighting

1. Introduction

We consider undirected simple graphs and denote by $V(G)$ and $E(G)$ the sets of vertices and edges of a graph G, respectively. We denote by $\deg_G(u)$, or simply $\deg(u)$ whenever the graph G is clear from the context,

the degree of the vertex u in G. An edge e_1 is at distance k from an edge e_2 if the shortest path containing e_1 and e_2 has $k+2$ edges.

An *edge colouring* of G is a mapping $\gamma : E(G) \longrightarrow \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. An edge colouring naturally induces a vertex colouring $\sigma_\gamma(v)$ given by

$$\sigma_\gamma(v) = \sum_{e \ni v} \gamma(e)$$

for every $v \in V(G)$. We will write $\sigma(v)$ instead of $\sigma_\gamma(v)$ whenever the edge colouring γ is clear from the context. We call the value $\gamma(e)$ the *colour of* e and the value $\sigma(u)$ the *sum at* u. A *neighbour-sum-distinguishing edge-colouring* of G is an edge colouring γ such that for every edge $uv \in E(G)$, $\sigma_\gamma(u) \neq \sigma_\gamma(v)$. Clearly, a graph admits a neighbour-sum-distinguishing edge-colouring if and only if it has no isolated edge.

The study of neighbour-sum-distinguishing edge-colouring of graphs was initiated by Karoński, Łuczak and Thomason [8]. They conjectured that every graph with no isolated edge admits a neighbour-sum-distinguishing edge-colouring with three colours. Despite many efforts to tackle it, this conjecture is still an open question. The best known result is due to Kalkowski, Karoński and Pfender [7], who proved that every graph with no isolated edge admits a neighbour-sum-distinguishing edge-colouring with five colours.

Several variants of the original problem have been introduced in the literature, such as neighbour-sum-distinguishing total-colouring [6, 9] or neighbour-sum-distinguishing arc-colouring [2, 3] for instance. Other variants have been obtained by defining the vertex colours by means of product, set or multiset of colours [1, 4, 5, 10].

In this paper we consider a game version of neighbour-sum-distinguishing edge-colouring. The *neighbour-sum-distinguishing edge-colouring game* on a graph G is a 2-player game where the two players, called Alice and Bob, alternately colour an uncoloured edge of G. Alice wins the game if, when all edges are coloured, the so-obtained edge colouring is a neighbour-sum-distinguishing edge-colouring of G. Therefore, Bob's goal is to produce an edge colouring such that two neighbouring vertices get the same sum, while Alice's goal is to prevent him from doing so. The neighbour-sum-distinguishing edge-colouring game on G with Alice having the first move will be referred to as the *A-game on* G. The neighbour-sum-distinguishing edge-colouring game on G with Bob having the first move will be referred to as the *B-game on* G.

The paper is organized as follows. In Section 2, we provide sufficient conditions on a graph G ensuring that Alice or Bob wins the A-game or the

B-game on G. We then consider the neighbour-sum-distinguishing edge-colouring game on paths, cycles, stars and double-stars in Section 3, complete graphs in Section 4 and complete bipartite graphs in Section 5.

2. General results

In this section we exhibit sufficient conditions on a graph G for Alice or Bob to have a winning strategy for the neighbour-sum-distinguishing edge-colouring game on G. The first two lemmas give sufficient conditions for Bob to win the A-game or the B-game.

A *balanced edge* in a graph G is an edge $uv \in E(G)$ with $\deg_G(u) = \deg_G(v)$.

Lemma 1. *Let G be a graph containing a balanced edge.*

1. *If $|E(G)|$ is even then Bob wins the A-game on G.*
2. *If $|E(G)|$ is odd then Bob wins the B-game on G.*

Proof. Let uv be a balanced edge in G with $\deg(u) = \deg(v) = d$ and $E_{u,v}$ be the set of edges having exactly one endpoint in $\{u, v\}$. We define Bob's strategy as follows:

- If Bob starts the game, he colours any edge from $E(G) \setminus E_{u,v}$ with any colour.
- If Alice plays in $E(G) \setminus E_{u,v}$, then Bob answers in $E(G) \setminus E_{u,v}$.
- If Alice assigns the colour c to an uncoloured edge uu', $u' \neq v$ (resp. vv', $v' \neq u$), then Bob assigns the colour c to some uncoloured edge vv', $v' \neq u$ (resp. uu', $u' \neq v$).

Since $|E_{u,v}|$ is even, $|E(G)|$ and $|E(G) \setminus E_{u,v}|$ have the same parity. Therefore, Bob can always apply this strategy in both games, so that we eventually get $\sigma(u) = \sigma(v)$. □

A *nice edge* in a graph G is an edge uv with $\deg_G(v) = 2$ and $\deg_G(w) \geq 2$, where w is the neighbour of v distinct from u. We denote by $|E(G)|_{nice}$ the number of nice edges in a graph G.

Lemma 2. *Let G be a connected graph.*

1. *If* $|E(G)|$ *is odd and* $|E(G)|_{nice} > \frac{1}{2}|E(G)|$ *then Bob wins the* B*-game on* G*.*
2. *If* $|E(G)|$ *is even and* $|E(G)|_{nice} > \frac{1}{2}|E(G)|$*, then Bob wins the* A*-game on* G*.*

PROOF. Bob's strategy consists in colouring a non-nice edge whenever possible. Doing so, Bob will be able to colour a nice edge on his last move whenever $|E(G)|_{nice} > \frac{1}{2}|E(G)|$. The parity of $|E(G)|$ implies that Bob's last move is the last move of the game. Let uv be the uncoloured nice edge to be coloured by Bob on his last move and let w be the neighbour of v distinct from u with $\deg(w) \geq 2$. Then the sum at w, say M, is strictly greater than the colour c of the edge vw. Therefore, Bob wins the game by colouring the edge uv with colour $M - c$. □

The last lemma of this section will give sufficient conditions for Alice to win the A-game or the B-game. We first introduce some definitions and notation.

A *pendant vertex* in a graph G is a vertex with $\deg_G(v) = 1$. For every vertex $v \in V(G)$, we denote by $\deg^p_G(v)$ the number of pendant neighbours of v. An *internal vertex* in a graph G is a vertex with $\deg_G(v) > 1$. A *pendant edge* in a graph G is an edge uv such that u or v is a pendant vertex. An *internal edge* in a graph G is an edge uv such that both vertices u and v are internal.

A *partial edge colouring* of a graph G is a mapping $\gamma : E_\gamma \longrightarrow \mathbb{N}^*$ where $E_\gamma \subseteq E(G)$ is the set of coloured edges of G. The graph G is said to be *partially coloured* by γ and the corresponding partially edge-coloured graph is denoted by (G, γ). A partial edge colouring γ is neighbour-sum-distinguishing if for every edge uv in G, $\sigma_\gamma(u) \neq \sigma_\gamma(v)$. A partial edge colouring γ with $E_\gamma = E(G)$ is an edge colouring of G.

Let (G, γ) be a partially edge-coloured graph. A vertex $v \in V(G)$ is *fully coloured* if all the edges incident with v are coloured. An edge $uv \in E(G)$ is *complete* if both vertices u and v are fully coloured, and *safe* if $\sigma_\gamma(u) \neq \sigma_\gamma(v)$. Observe that any complete pendant edge is necessarily safe.

Lemma 3. *Let* G *be a graph with no isolated edge and such that* $\deg^p_G(v) \geq \frac{1}{2}\deg_G(v) + 1$ *for every internal vertex* $v \in V(G)$*.*

(1) *If* $|E(G)|$ *is odd then Alice wins the* A*-game on* G*.*

(2) *If* $|E(G)|$ *is even then Alice wins the* B*-game on* G*.*

PROOF. First observe that if G is a union of disjoint stars, then Alice always wins the A-game and the B-game. Thus, we may assume that G contains at least one internal edge. Le us consider the A-game on G and let $|E(G)| = 2k + 1$. Let us denote by γ_i^A (resp. γ_i^B) the partial colouring of G obtained after Alice's i-th move, $1 \leq i \leq k + 1$ (resp. after Bob's i-th move, $1 \leq i \leq k$).

Consider the strategy for Alice given by the following rules:

(R1) On her first move, Alice colours any internal edge with any colour.

(R2) If Bob has coloured an internal edge, then Alice colours any internal edge with any colour, if possible.

(R3) If Bob has coloured the last uncoloured internal edge, then Alice chooses a vertex w incident with at least three uncoloured pendant edges and colours any of these uncoloured pendant edges with any colour.

(R4) If Bob has coloured a pendant edge uv with $\deg_G(u) = 1$, then Alice colours any internal edge incident with v with any colour, if possible.

(R5) If Bob has coloured a pendant edge uv with $\deg_G(u) = 1$, all internal edges incident with v are already coloured, and v is incident with at least two uncoloured pendant edges, then Alice colours any internal edge with any colour, if possible.

(R6) If Bob has coloured a pendant edge uv with $\deg_G(u) = 1$, all internal edges are already coloured, and v is incident with at least two uncoloured pendant edges, then Alice chooses a vertex w incident with at least three uncoloured pendant edges and colours any of these uncoloured pendant edges with any colour.

(R7) If Bob has coloured a pendant edge uv with $\deg_G(u) = 1$, and v is incident with only one uncoloured pendant edge $u'v$, then Alice colours the pendant edge $u'v$ with some colour c in such a way that every complete edge vv', if any, is safe.

We will prove that Alice can always apply this strategy (namely, that Rules (R3) and (R6) can be applied whenever needed), and that this strategy is a winning strategy for Alice. We first claim that after each of Alice's moves, the partially coloured graph (G, γ_i^A) satisfies the following properties:

(P1) Every complete edge is safe.

(P2) For every non fully coloured internal vertex v incident with at least one uncoloured internal edge, the number of coloured pendant edges incident with v is less than or equal to the number of coloured internal edges incident with v.

(P3) Every non fully coloured internal vertex v is incident with at least two uncoloured pendant edges.

We prove this claim by induction on i. The three properties clearly hold after Alice's first move. Suppose now that the partially coloured graphs $(G, \gamma_1^A), \ldots, (G, \gamma_p^A)$, $1 \leq p \leq k$, all satisfy the three properties and that Bob colours the edge xy on his p-th move.

Suppose first that xy is an internal edge. Since (G, γ_p^A) satisfies property (P2), xy is not a complete edge in (G, γ_p^B), and thus (G, γ_p^B) satisfies properties (P1),(P2) and (P3). According to her strategy, Alice will then apply either Rule (R2) or Rule (R3). If she applies Rule (R2), she colours an uncoloured internal edge and, similarly as for Bob's move, we get that (G, γ_{p+1}^A) satisfies the three properties. If all internal edges are coloured, since (G, γ_p^A) satisfies property (P2), and since $\deg_G^p(u) \geq \frac{1}{2}\deg_G(u) + 1$ and $\deg_G^p(v) \geq \frac{1}{2}\deg_G(v) + 1$, both vertices x and y are incident with at least three uncoloured pendant edges and thus Alice can apply Rule (R3). Again, (G, γ_{p+1}^A) clearly satisfies the three properties.

Suppose now that xy is a pendant edge, with $\deg_G(x) > 1$ and $\deg_G(y) = 1$. Alice will then apply one of the rules (R4), (R5), (R6) or (R7).

If Alice applies Rule (R4) then, since (G, γ_p^A) satisfies property (P2) and x was incident with an uncoloured internal edge in (G, γ_p^A), x was incident to at least three uncoloured pendant edges in (G, γ_p^A) and thus at least two in (G, γ_p^B). Hence (G, γ_{p+1}^A) satisfies property (P3). Since x is thus not fully coloured, (G, γ_{p+1}^A) also satisfies property (P1). Moreover, the numbers of pendant and internal coloured edges incident with x are both increased by one and thus, since (G, γ_p^A) satisfies property (P2), (G, γ_{p+1}^A) also satisfies (P2).

If Alice applies Rule (R5) then, since (G, γ_p^A) satisfies the three properties, both (G, γ_p^B) and (G, γ_{p+1}^A) also satisfy the three properties.

If all internal edges are coloured and after Bob's move x is incident with at least two uncoloured pendant edges then, since both (G, γ_p^A) and (G, γ_p^B) satisfy property (P3) and the number of uncoloured edges is odd (this follows from the fact that $|E(G)|$ is odd and the number of coloured

edges before Alice's move is even), there exists a vertex w incident with at least three uncoloured pendant edges, so that Alice can apply Rule (R6). Again, (G, γ^A_{p+1}) clearly satisfies the three properties.

Finally, if Alice applies Rule (R7), the choice of the colour c ensures that (G, γ^A_{p+1}) satisfies the three properties.

We thus get that the final edge colouring γ^A_{p+1} is neighbour-sum-distinguishing since the partially coloured graph (G, γ^A_{p+1}) satisfies property (P1).

We now consider the B-game on G. We claim that applying the same strategy as before (except Rule (R1) that is no longer valid), Alice wins the B-game on G. The proof is similar, up to the applicability of rules (R3) and (R6) which now follows from the fact that $|E(G)|$ is even and the number of coloured edges before any of Alice's moves is odd. □

In particular, Lemma 3 allows us to prove that Alice wins the A-game or the B-game on some special trees. A *caterpillar* is a tree T whose set of internal vertices induces a path, called the *central path* of T. We then have:

Corollary 4 (Special caterpillars). *Let T be a caterpillar, with central path $v_1v_2 \dots v_k$, such that $\deg_T(v_1) \geq 4$, $\deg_T(v_k) \geq 4$ and $\deg_T(v_i) \geq 6$ for every i, $2 \leq i \leq k-1$. We then have:*

(1) *If $|E(G)|$ is odd then Alice wins the A-game on G.*

(2) *If $|E(G)|$ is even then Alice wins the B-game on G.*

Proof. Note that since $\deg_T(v_1) \geq 4$, we have $\deg^p_G(v_1) = \deg_G(v_1) - 1 \geq \frac{1}{2}\deg_G(v_1) + 1$. Similarly, we have $\deg^p_G(v_k) = \deg_G(v_k) - 1 \geq \frac{1}{2}\deg_G(v_k) + 1$ and, since $\deg_T(v_i) \geq 6$, $\deg^p_G(v_i) = \deg_G(v_i) - 2 \geq \frac{1}{2}\deg_G(v_i) + 1$ for every i, $2 \leq i \leq k-1$. Therefore, T satisfies the conditions of Lemma 3 and the result follows. □

3. Simple graph classes

In this section we study the neighbour-sum-distinguishing edge-colouring game on simple classes of graphs, namely paths, cycles, stars and double-stars.

Theorem 5 (Paths and cycles). *Let P_n and C_n respectively denote the path and the cycle on n vertices, $n \geq 3$. We then have:*

1. *Bob wins the A-game on P_n if and only if $n \geq 5$.*
2. *Bob wins the B-game on P_n if and only if $n \notin \{3, 5\}$.*
3. *Bob wins the A-game on C_n for every n.*
4. *Bob wins the B-game on C_n if and only if $n \neq 4$.*

PROOF. Since every edge colouring of P_3 is a neighbour-sum-distinguishing edge-colouring, Alice wins both the A-game and the B-game on P_3.

Alice's strategy for the A-game on P_4 is to first colour the central edge of P_4 and then to colour the last edge with a colour distinct from the colour used by Bob.

Alice's strategy for the B-game on P_5 is to colour on her first move an edge at distance one from the edge previously coloured by Bob with a colour distinct from the colour used by Bob.

Similarly, Alice's strategy for the B-game on C_4 is to colour on her first move the edge at distance one from the edge previously coloured by Bob with a colour distinct from the colour used by Bob.

We now prove that Bob wins the game in all the remaining cases. Note that whenever two edges at distance one get the same colour, Bob wins the game. For the A-game on P_n, $n \geq 5$, Bob colours on his first move an edge at distance one from the edge previously coloured by Alice, with the same colour. For the B-game on P_4, Bob colours first the central edge and then the last edge with the same colour as the one used by Alice. For the B-game on P_n, $n \geq 6$, Bob colours first the third edge of P_n with some colour c. Alice then cannot prevent Bob from colouring on his second move either the first or the fifth edge with the same colour c. For the A-game on C_3, Bob colours an edge using the same colour as Alice on her first move. For the A-game on C_n, $n \geq 4$, Bob colours on his first move an edge at distance one from the edge coloured by Alice, using the same colour. For the B-game on C_3, Bob wins by colouring the last edge with the colour just used on the previous coloured edge. Finally, for the B-game on C_n, $n \geq 5$, Bob colours an edge with some colour c and Alice cannot prevent him from colouring with the same colour c an edge at distance one from the edge coloured first. □

Since every edge colouring of the star graph $K_{1,n}$, $n \geq 2$, is a neighbour-sum-distinguishing edge-colouring, we directly get the following:

Observation 6. *For every integer $n \geq 2$, Alice wins both the A-game and the B-game on $K_{1,n}$.*

The *double-star* $DS_{m,n}$, $m \geq n \geq 1$, is obtained from the two stars $K_{1,m}$ and $K_{1,n}$ by adding an edge joining their two centers. We prove the following:

Theorem 7 (Double-stars).

1. *For every integer $n \geq 1$, Bob wins the B-game on $DS_{n,n}$.*
2. *For every integer $n \geq 1$, Alice wins the A-game on $DS_{n,n}$.*
3. *For every integer $m > n \geq 1$, Alice wins the A-game on $DS_{m,n}$.*
4. *For every integer $m > n \geq 1$, Alice wins the B-game on $DS_{m,n}$.*

Proof. Let uv denote the central edge of the double-star $D_{m,n}$, with $\deg(u) = m + 1$ and $\deg(v) = n + 1$. Observe that only the sums at vertices u and v may be equal at the end of the game.

Since for every $n \geq 1$ the central edge of $DS_{n,n}$ is balanced, Theorem 7.1 directly follows from Lemma 1.

We now describe Alice's winning strategy for the A-game on $DS_{n,n}$. On her first move, she colours the central edge with any colour. Then, whenever Bob colours an edge incident with u (resp. v), she colours an edge incident with v (resp. u). On her last move, she colours the last edge with any colour ensuring that the sums at u and v are distinct.

Consider now the A-game on $D_{m,n}$, $m > n \geq 1$. Alice's winning strategy is as follows. On her first move, she colours the central edge with any colour. Then, whenever Bob colours with colour c an edge incident with u (resp. v), she colours with the same colour an edge incident with v (resp. u), if possible, otherwise she colours any remaining edge with any colour. Since the degrees of vertices u and v are distinct, the sums at u and v will necessarily be distinct at the end of the game.

Let us finally consider the B-game on $D_{m,n}$, $m > n \geq 1$. Alice's winning strategy is as follows. If Bob colours the central edge, then Alice colours any edge incident with u with any colour. If Bob colours with colour c an edge incident with u (resp. v), she colours with the same colour an edge incident with v (resp. u), if possible, otherwise she colours any remaining edge with any colour. Again, since the degrees of vertices u and v are distinct, the sums at u and v will necessarily be distinct at the end of the game.

This concludes the proof. □

4. Complete graphs

In this section we study the neighbour-sum-distinguishing edge-colouring game on complete graphs. We prove the following:

Theorem 8 (Complete graphs).

1. *For every integer $n \geq 3$, Bob wins the A-game on K_n.*
2. *For every integer $n \geq 3$, Bob wins the B-game on K_n if and only if $n \neq 4$.*

Proof. Since every edge in K_n is balanced and the number of edges of K_n is $\frac{n(n-1)}{2}$, we directly get by Lemma 1 that Bob wins the A-game on K_n whenever $n \equiv 0, 1 \pmod 4$ and that Bob wins the B-game on K_n whenever $n \equiv 2, 3 \pmod 4$.

We now consider the remaining cases. The proof of Theorem 8 will follow from a series of lemmas.

Lemma 9. *Bob wins the A-game on K_3.*

Proof. Bob simply colours the second edge with the colour used by Alice on her first move. □

Lemma 10. *Bob wins the A-game on K_4.*

Proof. We define Bob's strategy as follows: whenever Alice colours an uncoloured edge e with colour c, Bob colour the edge e' parallel with uv (that is, $e \cap e' = \emptyset$) with the same colour c. At the end of the game, all vertices clearly get the same sum. □

Lemma 11. *Alice wins the B-game on K_4.*

Proof. We define Alice's strategy as follows. On his first move, Bob colours some uncoloured edge e_1 with colour c_1. Alice then colours with the same colour c_1 the edge parallel with e_1. After this moment, all vertices clearly have the same sum and the uncoloured edges form C_4. Thus, Alice can apply her strategy for the B-game on C_4 described in Theorem 5. □

For larger values of n, we consider the structure of the graph induced by the uncoloured edges before some last rounds. It will be useful to introduce the following notation. We denote by R_k the graph induced by the k last remaining (uncoloured) edges. The next lemmas concern the graph R_4. This graph has four edges.

Remark 12. *Let us observe that in the cases we consider, the last even number of moves of the game are played first by Bob and then by Alice, alternately.*

In the following, we will say that we get equality on an edge uv whenever we get $\sigma(u) = \sigma(v)$.

Lemma 13. *If R_4 is a forest then Bob wins.*

Proof. Let us consider first the case where R_4 is a path of length four. Denote its vertices by x_1, x_2, x_3, x_4, x_5 and let $\sigma(x_i)$ be the sums at vertices x_i at this moment. On his first move, Bob colours the edge x_1x_2 with a huge (much greater than other values of the sums at vertices of R_4) colour H. Alice must colour the edge x_4x_5, since otherwise Bob in his next move would colour this edge in such a way that he would get the equality on x_1x_5. But then Bob colours the edge x_3x_4 by H' such that $\sigma(x_2)+H = \sigma(x_3)+H'$ and wins the game.

Now, let R_4 be a forest different from the path of length four. Then, R_4 has at least three leaves, u_1, u_2, u_3 say, belonging to three distinct edges e_1, e_2, e_3 say, respectively. Bob starts by colouring e_1 with a huge colour H, and after Alice's move, one of the above mentioned edges, say e_3, is still not coloured. Bob colours it with a colour H' in such a way that the equality $\sigma(u_1) + H = \sigma(u_3) + H'$ holds, where $\sigma(u_1), \sigma(u_3)$ denote the respective sums just before the four last moves. □

Lemma 14. *If R_4 is the graph $K_3 \cup K_2$ then Bob wins.*

Proof. Denote the vertices of R_4 as in Figure 1(a). On his first move, Bob colours the edge xy with a huge colour H.

- If Alice colours the edge yz (or xz), completing the colouring of edges incident to y (or x), then Bob colours u_1u_2 in such a way that he gets the equality on u_1y (or u_1x).

- If Alice colours the edge u_1u_2 then Bob colours xz in such a way that he gets the equality on yz. □

Lemma 15. *If $R_4 = T^+$ (see Figure 1(b)) and $\sigma(t) > \sigma(z)$, then Bob wins.*

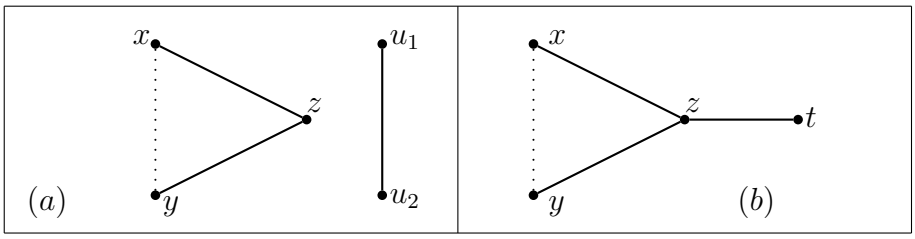

Figure 1: Some R_4 graphs: (a) $K_3 \cup K_2$ and (b) T^+. The first edge played by Bob is dotted.

PROOF. W.l.o.g. we may assume that $\sigma(x) \leq \sigma(y)$. Let $\sigma(x) = a, \sigma(y) = a + a'$, $\sigma(z) = b, \sigma(t) = b + b'$, where $b' > 0$. Bob colours the edge xy with a huge colour H. Alice must colour the edge zt, since otherwise Bob in his next move colours zt in such a way that either $\sigma(x) = \sigma(t)$ and all edges incident with x or t are already coloured or $\sigma(y) = \sigma(t)$ and all edges incident with y or t are already coloured, hence Bob would win. Thus, assume that Alice colours zt with M. After such moves we have

- $\sigma(x) = a + H$,
- $\sigma(y) = a + a' + H$,
- $\sigma(z) = b + M$,
- $\sigma(t) = b + b' + M$.

If $a + H < b + b' + M$, then Bob colours xz with $b + b' + M - (a + H)$. After his moves $\sigma(x) = \sigma(t)$ and all edges incident with x or t have been coloured and hence Bob wins. Thus, suppose that $a + H \geq b + b' + M$. Since $b' > 0$, the inequality $a + H \geq b + b' + M$ implies that $a + a' + H > b + M$. In such a case Bob colours xz with $a + a' + H - (b + M)$. After such a move, $\sigma(y) = \sigma(z)$ and yz is the only uncoloured edge incident with y or with z. Thus, no matter how Alice colours the last edge yz, after her move we still have $\sigma(y) = \sigma(z)$. □

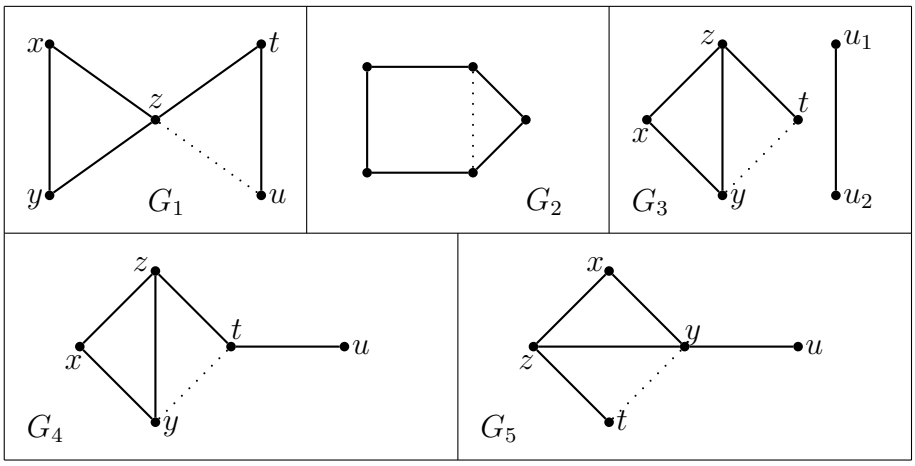

Figure 2: Some R_6 graphs. The first edge played by Bob is dotted.

Observation 16. *It is easy to see that if $R_4 = C_4$ then Bob has no good strategy. This is why he has to avoid $R_6 = K_{2,3}$ and $R_6 = K_4$. How to avoid the second situation is described in Lemma 18 where we consider the graphs R_8. In order to avoid $R_6 = K_{2,3}$ we have to avoid $R_8 = K_{2,4}$, $R_{10} = K_{2,5}$ and so on up to $R_{2n} = K_{2,n}$ (see Lemma 19). The way to avoid $R_{2n} = K_{2,n}$ is described below (after the proof of Lemma 19).*

Next lemma concerns the graphs R_6 containing the graph T^+, defined in Figure 1(b), as a subgraph. These graphs have six edges.

Lemma 17. *If $R_6 \neq K_4$ and $R_6 \neq K_{2,3}$ then Bob wins.*

Proof. If R_6 contains at most one cycle then Bob in his first move destroys this cycle and, regardless of Alice's move, the graph R_4 is a forest. Thus, we can apply Lemma 13.

If R_6 contains more than one cycle, then it is isomorphic to one of the five graphs drawn in Figure 2 or $R_6 = C_3 \cup C_3$. We consider six cases.

Case 1. $R_6 = G_1$. Then Bob plays a huge H on zu. If Alice colours tu, in order to get $R_4 = T^+$, Bob colours zt in such a way that he gets the equality on tu. If Alice colours an edge different from tu, the graph R_4 is either the forest or the graph $K_3 \cup K_2$ and we can apply either Lemma 13 or Lemma 14.

Case 2. $R_6 = G_2$. Then Bob colours the chord of the cycle C_5. So, after Alice's move, we get the path of length four and we apply again Lemma 13.

Case 3. $R_6 = G_3$. Then Bob plays a huge H on ty. If Alice colours u_1u_2, she gets $R_4 = T^+$, but with $\sigma(t)$ much greater than $\sigma(z)$ and we can apply Lemma 15. If Alice colours an edge different from u_1u_2, the graph R_4 is either a forest or the graph $K_3 \cup K_2$ and we can apply either Lemma 13 or Lemma 14.
Case 4. $R_6 = G_4$. Then Bob plays a huge H on ty. In order to get $R_4 = T^+$, Alice has to colour tu. Then, she gets $R_4 = T^+$, but with $\sigma(t)$ much greater than $\sigma(z)$ and we can apply Lemma 15. If Alice colours an edge different from tu, the graph R_4 is either the forest or the graph $K_3 \cup K_2$ and we can apply either Lemma 13 or Lemma 14.
Case 5. $R_6 = G_5$. Let $\sigma(v)$ denote the sum at a vertex v at this moment. Then Bob plays a huge H on ty. In order to get $R_4 = T^+$, Alice can colour either yu or zt.

- If Alice colours zt, say by M, Bob plays on yu in such a way that the equality on tu holds. It is possible since the present value at t, equal to $\sigma(t) + H + M$, is much bigger than $\sigma(u)$.

- If Alice colours yu, say by M, she gets $R_4 = T^+$, but with $\sigma(t) + H$ at the vertex t much bigger than $\sigma(z)$ and we can apply Lemma 15.

Case 6. $R_6 = C_3 \cup C_3$. After first moves of Alice and Bob, we get R_4 isomorphic either to a forest or to the graph $K_3 \cup K_2$, so we shall surely be able to use either Lemma 13 or Lemma 14. □

The next lemma concerns the graphs R_8 containing the graph K_4 as a subgraph.

Lemma 18. *If R_8 contains the graph K_4 as a subgraph then Bob wins.*

Proof. Denote by $\{u_1, u_2, u_3, u_4\}$ the vertices of the graph K_4. Consider first the case when R_8 contains one more vertex, say u_5 joined with u_2 and u_3. Bob starts by colouring the edge u_1u_3.

If Alice plays on one of the following edges: u_1u_2, u_1u_4, u_5u_2, u_5u_3, then R_6 is either G_4 or G_5 from Lemma 17 and Bob wins.

If Alice plays on one of the two edges u_2u_3 or u_2u_4, then Bob plays on the remaining one and R_5 is the cycle C_5. In the next move Alice has to create the path P_5.

If Alice plays on u_3u_4 then we get G_1, from Lemma 17 and Bob wins.

Consider now the case when none of the two edges of R_8 which are not edges of K_4 is on a cycle. Denote these edges by e_1, e_2. Bob colours one of the edges of K_4.

If Alice colours another edge of K_4 then Bob is able to destroy all cycles in R_5 in his next move. So, Alice has to colour either e_1 or e_2, but then we get one of the graphs from Lemma 17 and Bob wins. □

Lemma 19. *Let $k \geq 3$. If $R_{2k+2} \neq K_{2,k+1}$ then Bob is able to get $R_{2k} \neq K_{2,k}$.*

Proof. Let $G = R_{2k+2}$. Without loss of generality we may assume that G is a graph obtained from $K_{2,k}$ by adding two edges and, maybe, also some vertices. By G' (G'') we denote the graph obtained from G by removing one (two) edges from G, respectively.

If $k \geq 5$ then G contains exactly two vertices, x_1, x_2 say, of degree greater than or equal to k and at least one vertex, say y_1, of degree two and adjacent to both vertices x_1 and x_2. If Bob colours the edge x_1y_1, the vertex y_1 becomes of degree one, and, since $K_{2,k}$ does not contain leaves, Alice has to colour the edge x_2y_1. If afterwards the graph $G'' = K_{2,k}$ then $G = K_{2,k+2}$, a contradiction.

Let $k = 4$. Let us suppose first that G contains exactly two vertices, x_1, x_2 say, of degree greater than or equal to 4. If G has at least one vertex, say y_1, of degree two, adjacent to both vertices x_1 and x_2 we proceed as above. If there is no such vertex, the vertices x_1 and x_2 are of degree exactly four. If Bob colours one of the edges adjacent do x_1, then G' has only one vertex of degree 4, and $G'' = K_{2,k}$ is impossible to be obtained.

Similarly, in the case when G contains more than two vertices of degree greater than or equal to 4, then G contains exactly three such vertices and all of them are of degree four. Moreover, Bob may colour one of the edges connecting two of these vertices, and thus again we get G' with only one vertex of degree 4.

A straightforward but somewhat tedious analysis of the case $k = 3$ is left to the reader. □

We are now able to complete the proof of Theorem 8.

First, we shall show that Bob is able to avoid the graph $R_{2(n-2)} = K_{2,n-2}$. Consider the A-game on K_n with $n \equiv 2, 3 \pmod 4$. Then $\frac{n(n-1)}{2}$ is odd. The number of moves before the *critical situation* (where only $2(n-2)$ uncoloured edges remain) is equal to

$$\frac{n(n-1)}{2} - 2(n-2) = \frac{(n-2)(n-3)}{2} + 1$$

which is also odd. If $R_{2(n-2)} = K_{2,n-2}$ then the coloured edges should form the graph $K_{n-2} \cup K_2$. The longest path in this graph is of order $n-2$. Therefore, the aim of Bob is to colour a path of order at least $n-1$ before the critical situation is attained.

Since Alice starts, Bob has $\frac{(n-2)(n-3)}{4}$ moves. Observe that the first edge coloured by Alice can be used by Bob to build the path. Therefore, he needs only $n-3$ moves.

We have

$$\frac{(n-2)(n-3)}{4} \geq n-3$$

for $n \geq 6$.

In the case of the B-game, with $n \equiv 0, 1 \pmod 4$, the number $\frac{n(n-1)}{2}$ is even. The number of moves until the critical situation has been reached is equal to $\frac{n(n-1)}{2} - 2(n-2) = \frac{(n-2)(n-3)}{2} + 1$, which is also even. Since Bob starts, he has $\frac{n^2-5n+8}{4}$ moves. Observe that also in this case the first edge coloured by Alice can be used by Bob to build the path. Therefore, he needs only $n-3$ moves.

We have

$$\frac{n^2-5n+8}{4} \geq n-3$$

for $n \geq 5$.

We define Bob's strategy as follows:

- Until a situation where only $2(n-2)$ uncoloured edges remain is reached, Bob builds a path as long as possible. As we showed above, this strategy allows to avoid the graph $R_{2(n-2)} = K_{2,n-2}$.
- Next, Bob continues avoiding $R_{2k} = K_{2,k}$ (cf. Lemma 19) until the moment when only eight moves remain.
- If R_8 contains K_4, then Bob plays as in Lemma 18. If not, he continues his strategy and avoids $K_{2,3}$ in the next round.
- Since R_6 is neither $K_{2,3}$ nor K_4, Bob continues the game using strategy described in Lemma 17.

So, Bob wins in every case. □

5. Complete bipartite graphs

In this section we study the neighbour-sum-distinguishing edge-colouring game on complete bipartite graphs of the form $K_{2,n}$, $n \geq 2$. We prove the following:

Theorem 20 (Complete bipartite graphs).

1. *For every integer $n \geq 2$, Bob wins the A-game on $K_{2,n}$.*
2. *For every integer $n \geq 2$, Alice wins the B-game on $K_{2,n}$.*

Proof. Each edge of $K_{2,n}$ is nice and $|E(K_{2,n})|$ is even, then Lemma 2 implies that Bob wins the A-game.

Let us denote by $\{u, v\} \cup \{x_1, \ldots, x_n\}$ the bipartition of $V(K_{2,n})$. By Theorem 5, we know that Alice wins the B-game on $K_{2,2}$. We can thus assume that $n \geq 3$. We describe Alice's strategy for the B-game on $K_{2,n}$. Assume that on his i-th move Bob colours with colour c_i the edge wx_i, $w \in \{u, v\}$, $1 \leq i \leq n$. Alice then responds by colouring with the same colour c_i an edge $w'x_{i'}$ such that $i' \neq i$ and $\{w, w'\} = \{u, v\}$ (note that this is always possible since Bob plays first). Alice has to only be careful while choosing her last but one move, and avoid leaving two uncoloured adjacent edges afterwards. At the end of the game, the sum at both vertices u and v is thus $\sum_{i=1}^{n} c_i$ and the sum at any vertex x_i is $c_i + c_{i'}$ with $i' \neq i$. Therefore, since $n \geq 3$, the sum at vertices u and v is strictly greater than the sum at vertex x_i for every i, $1 \leq i \leq n$, and thus Alice wins the game. □

We leave as an open problem the question of determining who wins the A-game or the B-game on general complete bipartite graphs $K_{m,n}$, $3 \leq m \leq n$.

Acknowledgement

Research contained in this paper were supported by the joint program PICS CNRS 6367 "Graph partitions". Most of this work has been done while O. Baudon, E. Sidorowicz and É. Sopena were visiting AGH University of Science and Technology, and while J. Przybyło and M. Woźniak were visiting LaBRI. The third author was financed within the program of the Polish Minister of Science and Higher Education named "Iuventus Plus" in years 2015-2017, project no. IP2014 038873. The sixth author was supported by the National Science Centre, Poland, grant no. DEC-2013/09/B/ST1/01772. The third and sixth authors were also partly supported by the Polish Ministry of Science and Higher Education.

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