

Neighbour distinguishing edge-decompositions of graphs

Mohammed SENHAJI

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Abstract

Can we weight the edges of a graph G , with weights 1, 2, and 3, such that each pair of adjacent vertices of G are distinguished by the sum of their incident weights ? This question, introduced by Karoński, Łuczak and Thomason [KLT04], later becomes the famous 1-2-3 Conjecture. Formally, for a positive integer k , a k -edge-weighting ω of a graph G is a mapping from $E(G)$ to $\{1, \dots, k\}$. If the induced vertex sum colouring σ_ω , defined, for every vertex v of G , by $\sigma_\omega(v) = \sum_{e \in E(G), e \ni v} \omega(e)$, is a proper colouring,

then ω is a neighbour-sum-distinguishing (or nsd for short) edge-weighting of G . The smallest integer k , such that G admits an nsd k -edge-weighting is the neighbour-sum-distinguishing index of G denoted by $\chi_\Sigma^e(G)$. Note that the ends of an isolated edge can not be sum-distinguished. Therefore nsd edge weightings are considered only for graphs with no isolated edges. Such graphs are called nice graphs. With this notation the 1-2-3 Conjecture can be formulated as follows:

1-2-3 Conjecture. *For every nice graph G , $\chi_\Sigma^e(G) \leq 3$.*

The problem of nsd edge-weightings arose as a local version of the irregularity strength of graphs introduced in [CEO88], that is, for a graph G , the maximal weight used in an edge-weighting of G , such that every two vertices (adjacent or not) are sum-distinguished.

In 2010 Kalkowski, Karoński and Pfender [KKP10] proved that every nice graph admits an nsd 5-edge-weighting. But even before that, many other authors tried to make progress on the 1-2-3 Conjecture (see [ABDM⁺07], [ABDR08] and [WY08]). In 2011, Dudek and Wajc [DW11] proved that the problem of deciding if a graph admits an nsd 2-edge-weighting is NP-Complete. However, in 2016, Thomassen, Wu et Zhang [TWZ16] proved that the problem is polynomial on bipartite graphs.

Toward the 1-2-3 Conjecture, Przybyło and Woźniak [PW10] introduced nsd total-weightings where we assign weights to edges and vertices of a graph, and require adjacent vertices to be distinguished by the sum of their incident weights and their own weight. Przybyło and Woźniak conjectured in the same paper that every nice graph admits an nsd total-weighting using weights 1 and 2. In 2010, Kalkowski [Kal10] proved that every graph admits an nsd total 3-weighting,

Note that if a graph is locally irregular, that is, with no adjacent vertices of the same degree, then a 1-edge-weighting is neighbour-sum-distinguishing. As assigning positive

weights to the edges of a graph is equivalent to duplicating these edges from the perspective of weighted degree, another way to see nsd edge-weightings is to ask, for a graph G , how many copies of each edge of G must be added to G in order to turn it into a locally irregular multigraph?. This motivated Baudon, Bensmail, Przybyło and Woźniak to introduce, in [BBPW15], the locally irregular decompositions, that is, partitions of the edges of a graph into sets, each inducing a locally irregular graph. As for nsd edge-weighting, not every graph admits such a decomposition: there exist exceptions, fully characterized by Bensmail in the same paper. Graphs that are not exceptions are decomposable. Moreover, the authors stated the following conjecture:

Conjecture 1 ([BBPW15]). *Every decomposable graph G can be decomposed into at most three locally irregular subgraphs.*

In this thesis we explore several variants of the 1-2-3 Conjecture, and their links with locally irregular decompositions.

Equitable neighbour-sum-distinguishing weightings

For a positive integer k , a k -edge-weighting or total k -weighting of a graph G is equitable if every weight in $\{1, \dots, k\}$ is used on the same number of elements (up to a difference of 1). The smallest integer k , such that G admits an nsd k -edge-weighting (resp. total k -weighting) is denoted by $\overline{\chi}_\Sigma^e(G)$ (resp. $\overline{\chi}_\Sigma^t(G)$). We explore the value of $\overline{\chi}_\Sigma^e$ for complete graphs, complete bipartite graphs, forests and several families of sparse graphs (*e.g.* cycles and Θ -graphs). The main goal is to find families of graphs for which $\overline{\chi}_\Sigma^e > \chi_\Sigma^e$. However, except for some small graphs, namely $K_{3,3}$, K_4 and three cubic bipartite graphs on 10 and 14 vertices, we did not manage to come up with examples of such graphs. Moreover, among the graphs we study, only K_4 requires weights strictly greater than 3 in an equitable nsd edge-weighting. While this could motivate us to conjecture that $\overline{\chi}_\Sigma^e(G) \leq 3$ for every connected nice graph different from K_4 , we prefer to wait for further results to propose such a conjecture. Moreover, in the general case, except for an obvious exponential upper bound of $2^{|E(G)|-1}$ obtained by assigning a distinct power of 2 to each edge, we did not succeed to exhibit a small upper bound on $\overline{\chi}_\Sigma^e$ in general, and we still do not know if it is bounded. On the other hand we also determine the value of $\overline{\chi}_\Sigma^t$ for complete graphs and bipartite graphs. The obtained results were published in [BPP⁺17].

Injective neighbour-sum-distinguishing edge-weightings

An injective edge-weighting of a graph is an edge-weighting where every two edges get different weights. We get interested in injective nsd weightings for two main reasons. First, every injective edge-weighting is an equitable one, hence, every upper bound obtained on injective edge-weightings is also an upper bound on $\overline{\chi}_\Sigma^e$. Second, in the optimal case, that is when we use weights in $\{1, \dots, |E(G)|\}$ on the edges of a graph G , injective nsd edge-weighting is a local variant of the antimagic labelling of graphs [HR90]. We denote by $\chi_\Sigma^{e,1}(G)$, the smallest positive integer k such that G admits an injective nsd k -edge-weighting. We prove that $\chi_\Sigma^{e,1}(G) = |E(G)|$ if G is a forest and exhibit two upper bounds on $\chi_\Sigma^{e,1}$, namely $|E(G)| + 2\Delta(G)$ and $2 \cdot |E(G)|$ in general case. Moreover we prove upper bounds of the form $|E(G)| + c$ for a small constant c in the cases of 2-degenerate graphs and graphs with maximum average degree at most three. The obtained results were published in [BSSL17], where we also propose the local version of the antimagic labelling conjecture [HR90]:

Conjecture 2. *For every nice graph G , $\chi_{\Sigma}^{e,1}(G) = |E(G)|$.*

This conjecture was later proved by Haslegrave in [Has17].

Locally irregular decompositions of subcubic graphs

For a decomposable graph G , we denote by $\chi'_{\text{irr}}(G)$, the smallest number of classes in a locally irregular decomposition of G . Toward Conjecture 1, we study the value of $\chi'_{\text{irr}}(G)$ for subcubic graphs and prove the following theorem:

Theorem 3. *For every decomposable subcubic graph G , $\chi'_{\text{irr}}(G) \leq 5$. Moreover if $\text{mad}(G) \leq \frac{12}{5}$, then $\chi'_{\text{irr}}(G) \leq 3$.*

We also study two relaxations of Conjecture 1. In the first one we allow decomposition classes to induce a disjoint union of a locally irregular subgraph and a matching (a set of independent edges). And in the second one we generalize that by allowing each decomposition class to induce a disjoint union of a locally irregular graph and a regular graph. In both cases we completely determine the minimum number of classes in such decompositions for subcubic graphs.

The obtained results were published in [BBH⁺18].

Other variants

We consider other variants of nsd edge-weightings. The first variant requires that the induced sums on adjacent vertices differ by at least 2. Such a weighting is a neighbour-sum-2-distinguishing (ns2d for short) edge-weighting, and the smallest integer k such that a graph G admits an ns2d k -edge-weighting is denoted by $\chi_{\Sigma>1}^e(G)$. We prove that the problem of determining the value of $\chi_{\Sigma>1}^e$ is NP-complete in general, even when restricted to planar bipartite 3-degenerate graphs. This contrasts with the fact that the problem of determining $\chi_{\Sigma}^e(G)$ is polynomial if G is bipartite. However, the problem becomes polynomial on trees. Moreover, following some observations on links with the 1-2-3 Conjecture, especially for bipartite graphs, we propose the following conjecture:

Conjecture 4. *Every nice graph G admit an ns2d edge-weighting with the weights 1,3 and 5, and thus, $\chi_{\Sigma>1}^e(G) \leq 5$.*

The obtained results were submitted in a paper which has been accepted for publication in *Discrete Applied Mathematics*.

We also study a game variant of nsd edge-weightings, that is a game with two players, Alice and Bob, weighting alternately one edge of their choice at each turn, using positive integers. Alice wins if the final weighting is nsd, otherwise Bob wins. We completely determine the game issue on complete graphs and complete bipartite graphs, and define several sufficient conditions for Alice or Bob to win the game.

The obtained results were published in [BPS⁺17].

A general decomposition theory

Finally we introduce a general theory of graph decomposition, generalising both nsd edge-weighting and locally irregular decompositions. The idea is to assign a colour to each edge in addition of a weight. This allows us to calculate for each vertex a tuple of induced coloured sums. Formally, if $\ell, k \geq 1$ are two integers, and G is a graph, then to each

edge of G , we assign, via an edge-colouring ω , a pair (α, β) , where $\alpha \in \{1, \dots, \ell\}$ and $\beta \in \{1, \dots, k\}$, which can be regarded as a *coloured weight* (with value β and colour α). Now, for every vertex v of G , and every colour $\alpha \in \{1, \dots, \ell\}$, one can compute the *weighted α -degree* $\sigma_\alpha(v)$ of v , being the sum of weights with colour α incident to v . So, with every vertex v is associated a palette $(\sigma_1(v), \dots, \sigma_\ell(v))$ of ℓ coloured sums.

When dealing with these new notions, there are many possible ways for asking for distinction, as several coloured sums are available. In this work, we focus on the following three distinction variants. Every two adjacent vertices u and v of G are considered *weakly distinguished* if there is an $\alpha \in \{1, \dots, \ell\}$ such that $\sigma_\alpha(u) \neq \sigma_\alpha(v)$. Vertices u and v are considered *standardly distinguished* if, assuming $\omega(uv) = (\alpha, \beta)$, we have $\sigma_\alpha(u) \neq \sigma_\alpha(v)$. Finally u and v are considered *strongly distinguished* if, for every $\alpha \in \{1, \dots, \ell\}$, we have $\sigma_\alpha(u) = \sigma_\alpha(v) = 0$, or $\sigma_\alpha(u) \neq \sigma_\alpha(v)$. Assuming ω verifies one of the weak, standard or strong distinction condition for every pair of adjacent vertices, we say that ω is a *weak, standard or strong (ℓ, k) -edge-colouring*, respectively. We also say that G is *weakly, standardly or strongly (ℓ, k) -edge-coloured*, respectively. Finally, we say that G is *weakly, standardly or strongly (ℓ, k) -edge-colourable*, if there are $\ell', k' \geq 1$ with $\ell' \leq \ell$ and $k' \leq k$ such that G can be weakly, standardly or strongly (ℓ', k') -edge-coloured, respectively.

Rephrased with this new terminology, the 1-2-3 Conjecture asks whether every nice graph is strongly $(1, 3)$ -edge-colourable. Similarly, Conjecture 1 asks whether every decomposable graph is standardly $(3, 1)$ -edge-colourable. As a leading objective, we consider the following conjecture, which has flavour of both problems:

Conjecture 5. *Every graph with no isolated edge or triangle is strongly $(2, 2)$ -edge-colourable.*

We prove Conjecture 5 for nice complete graphs and bipartite graphs, and we characterize the bipartite graphs that do not admit strong $(1, 1)$ -, $(1, 2)$ - or $(2, 1)$ -edge-colourings, using results proved by Thomassen, Wu and Zhang in [TWZ16], which characterise nice bipartite graphs that do not admit an nsd 2-edge-weighting.

Then we explore the standard version of Conjecture 5. We prove it for nice 2-degenerate graphs and graphs with maximum degree 3. We also prove that every nice graph is standardly $(40, 3)$ -edge-colourable, and every nice 9-colourable graph is standardly $(2, 3)$ -edge-colourable.

Finally we study the weak version of Conjecture 5, proving that every nice graph is weakly $(3, 2)$ -edge-colourable.

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