

# The Weak $(2,2)$ -Labelling Problem for graphs with forbidden induced structures

Julien Bensmail, Hervé Hocquard, Pierre-Marie Marcille

LaBRI, Université de Bordeaux

CALDAM 2023

# Labellings and $i$ -sums

## Definition

An  $(\alpha, \beta)$ -labelling of a graph  $G$  is a function  $\ell : E(G) \rightarrow \{1, \dots, \alpha\} \times \{1, \dots, \beta\}$ , where the label is a couple of a color and a value.

## Definition

The  $i$ -sums of a vertex  $u$ , denoted  $\sigma_i(u)$ , is the sum of the values of its incident edges labelled with color  $i$ . Formally,

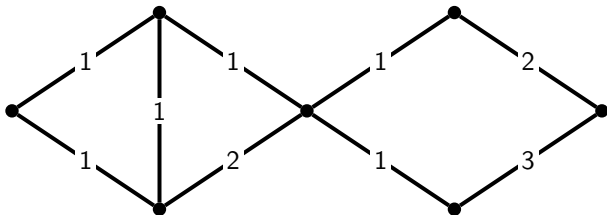
$$\sigma_i(u) = \sum_{\substack{e \in I(u) \\ \ell(e)[0]=i}} \ell(e)[1]$$

where  $I(u)$  is the set of edges incident to  $u$ .

# Distinguishing labellings

## Definition

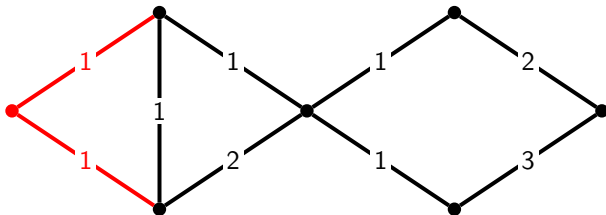
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

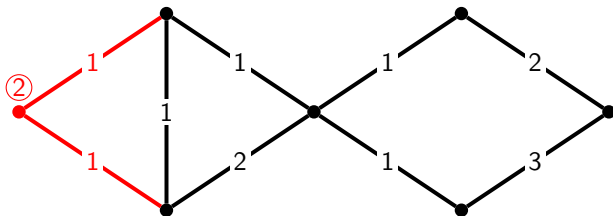
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

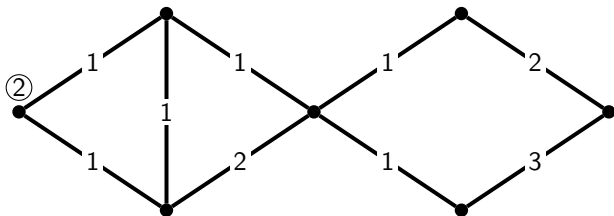
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

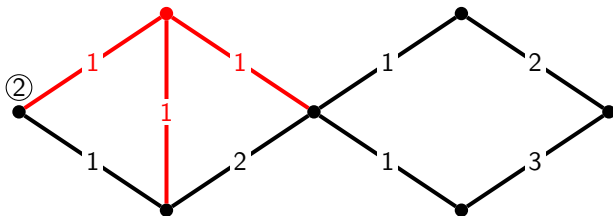
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

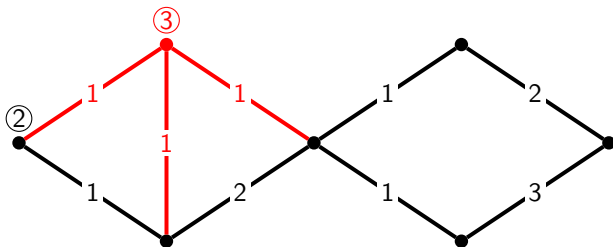
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.

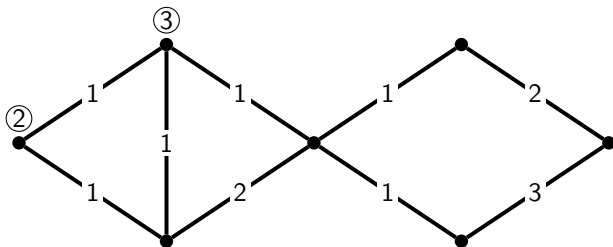




# Distinguishing labellings

## Definition

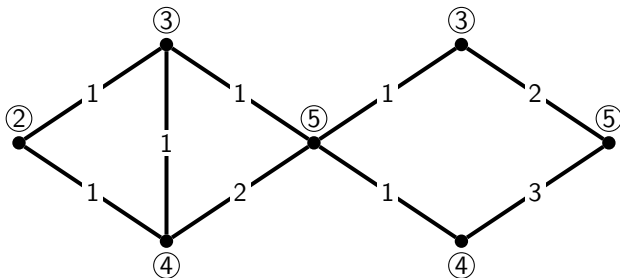
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

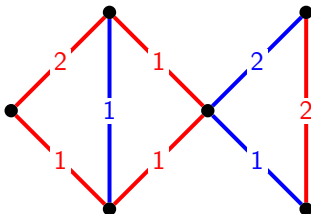
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

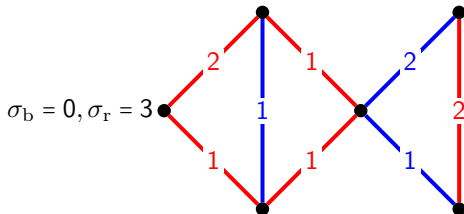
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

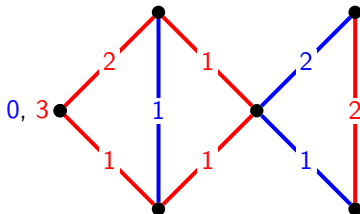
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

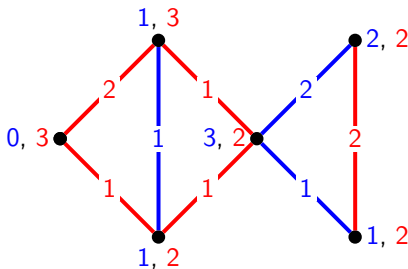
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

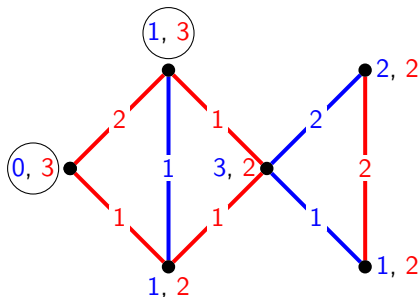
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

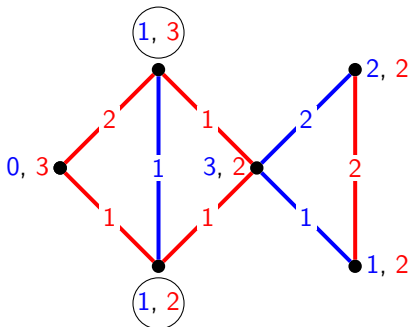
We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.



# Distinguishing labellings

## Definition

We said it is distinguishing if for every two adjacent vertices  $u$  and  $v$  of  $G$ , there is an  $i \in \{1, \dots, \alpha\}$  such that the  $i$ -sum of  $u$  and  $v$  differs.





# Conjectures

There are a few conjectures about "all" graphs admitting an  $(\alpha, \beta)$ -labelling.

1-2-3 Conjecture (Karonski *et al.*, 2004)

All graphs admit a  $(1, 3)$ -labelling.

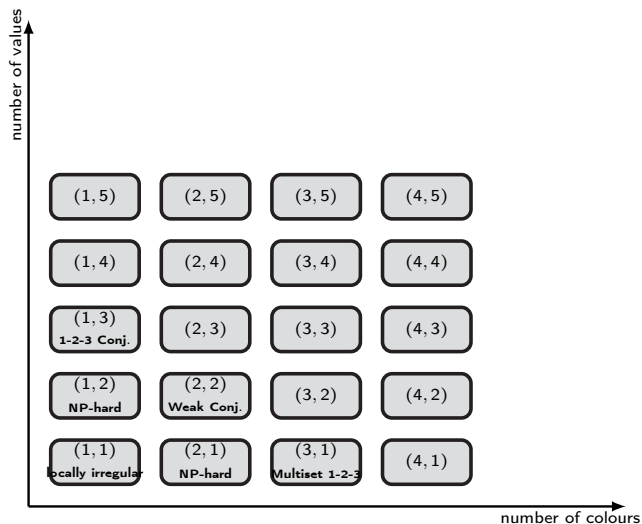
$(2, 2)$ -Conjecture (Baudon *et al.*, 2019)

All graphs admit a  $(2, 2)$ -labelling.

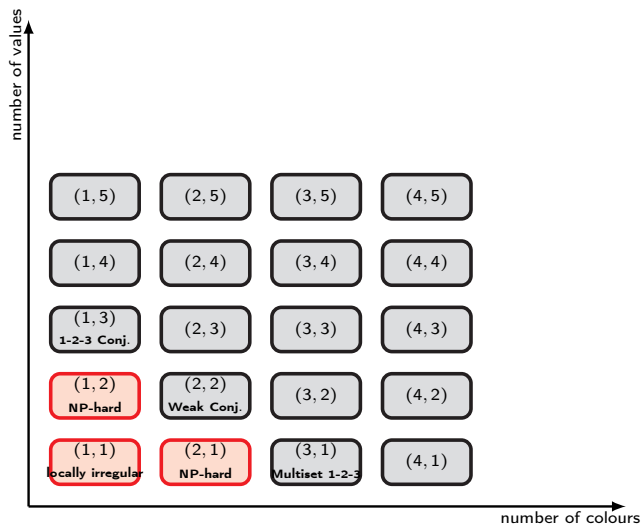
## Observation

The 1-2-3 Conjecture implies the  $(2, 2)$ -Conjecture.

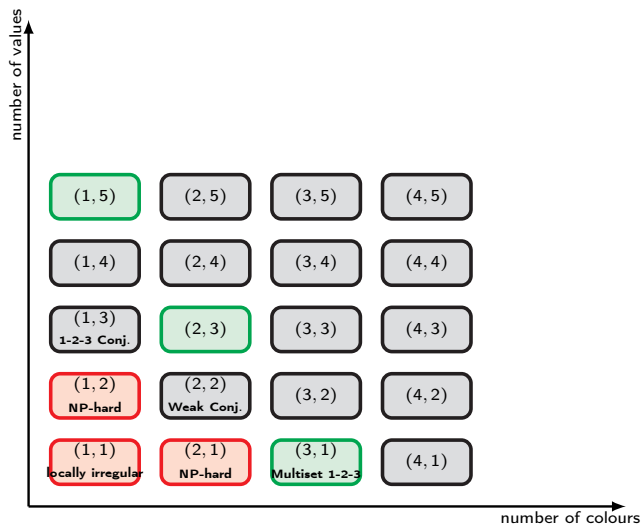
# Current knowledge



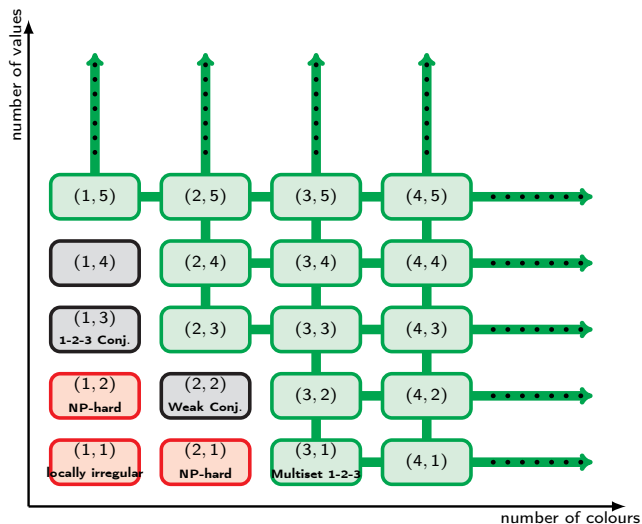
# Current knowledge



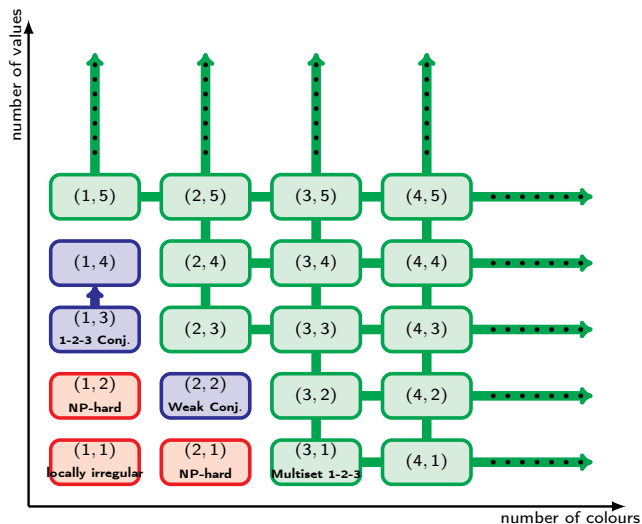
# Current knowledge



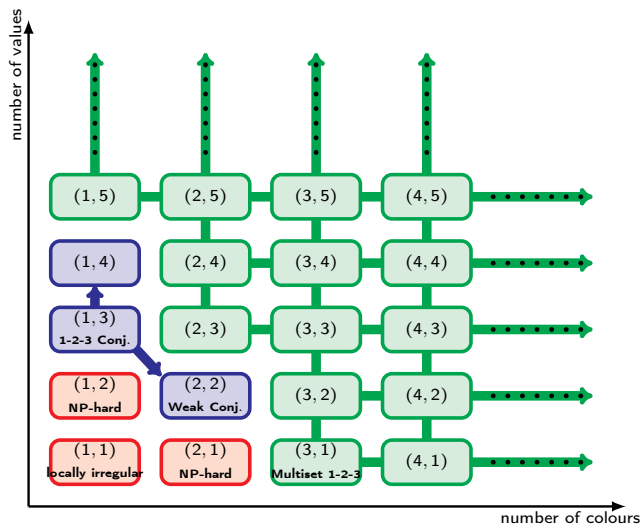
# Current knowledge



# Current knowledge



# Current knowledge

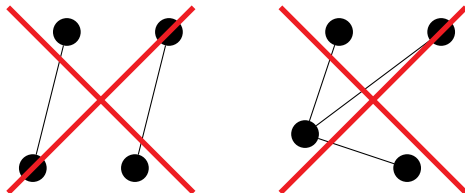


# Some known cases and our contribution

- The 1-2-3 Conjecture is known to hold for 3-colorable graphs.
- Hence, the  $(2,2)$ -Conjecture holds for 3-colorable graphs.
- The  $(2,2)$ -Conjecture in fact holds for 4-colorable graphs.

Theorem (Bensmail, Hocquard, M., 2022+)

The  $(2,2)$ -Conjecture holds for  $2K_2$ -free graphs and  $K_{1,3}$ -free graphs.





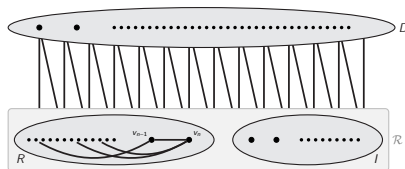
# $2K_2$ -free graphs

We want to show the  $(2,2)$ -Conjecture for those graphs. Remember it holds for 4-colorable graphs.

## Theorem

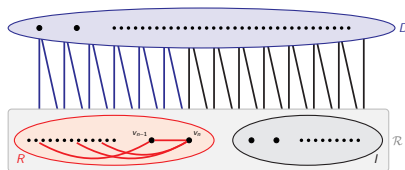
Every  $2K_2$ -free graph with chromatic number 5 admits a distinguishing  $(2,2)$ -labelling.

# Outline of the proof



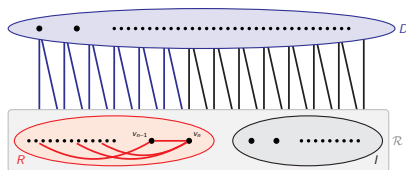
Isolate a maximum independent set to deal with a 4-colorable subgraph.

# Outline of the proof



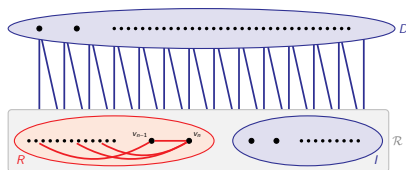
Use the 4-coloring to compute a  $(2, 2)$ -labelling.

# Outline of the proof



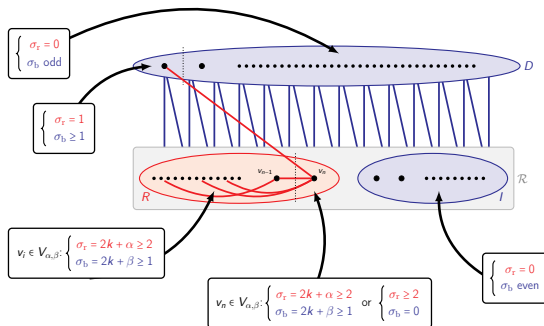
Ensure those vertices have a large red sum.

# Outline of the proof



Label the remaining edges with mostly blue labels.

# Outline of the proof



We labelled each edge so that every subset of the partition has these distinguishing properties.

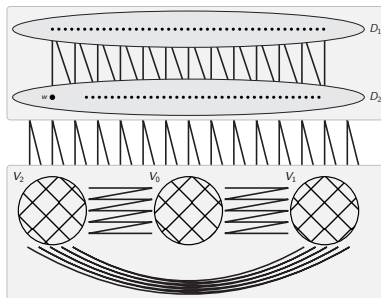
# $2K_2$ -free graphs of high chromatic number

## Theorem

Every  $2K_2$ -free graph with chromatic number at least 6 admits a distinguishing  $(2, 2)$ -labelling.

The proof is very similar...

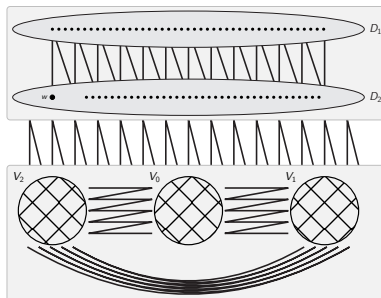
# Outline of the proof



Separate a maximum independent set of the graph.

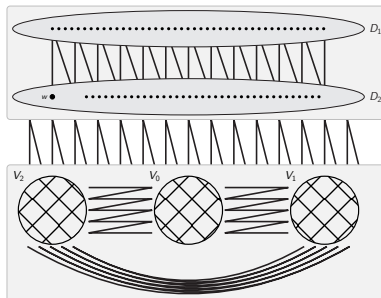


# Outline of the proof



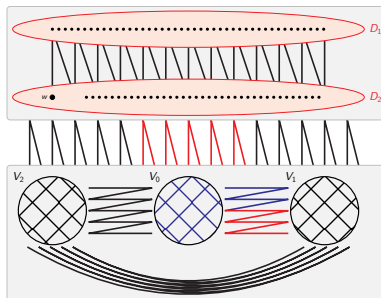
Separate another maximum independent set of the remaining subgraph.

# Outline of the proof



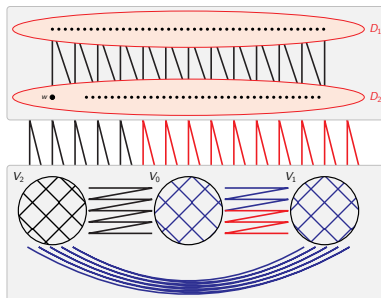
A tri-partition is a partition of a 4-colorable graph into 3 subsets  $V_0$ ,  $V_1$ ,  $V_2$  such that every vertex has at least one neighbor in the next set, and more neighbors in the next set than in its own set.

# Outline of the proof



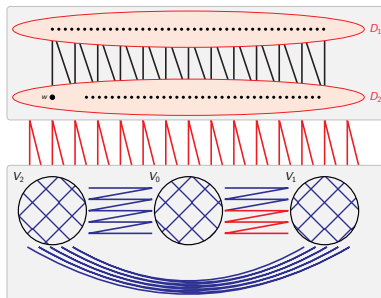
Label each one of those subgraphs and the edges between them.

# Outline of the proof



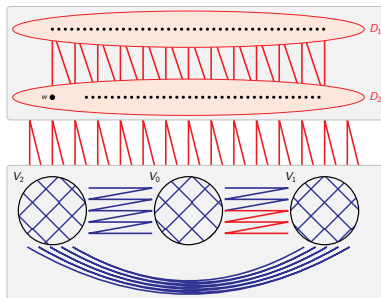
Label each one of those subgraphs and the edges between them.

# Outline of the proof



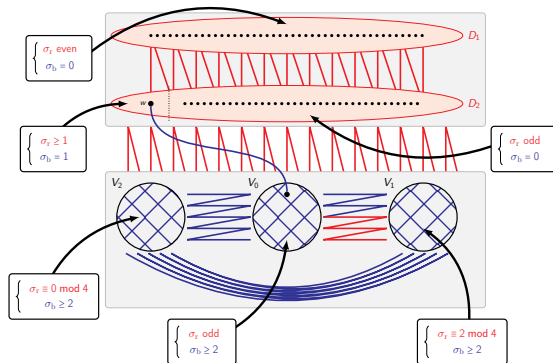
Label each one of those subgraphs and the edges between them.

# Outline of the proof



Label each one of those subgraphs and the edges between them.

# Outline of the proof



Verify some properties that ensure the distinguishing result.

# Claw-free graphs

We also prove a similar result for  $K_{1,3}$ -free graphs.

## Theorem

The  $(2, 2)$ -Conjecture holds for  $K_{1,3}$ -free graphs.

The proof is very similar, but more technical:

- Each set can now have multiple connected components with edges.
- In the case of large chromatic number, the two independent sets can have multiple connected components.



# Perspectives and conclusion

A few perspectives :

- Proving the  $(2,2)$ -Conjecture for other graph classes with forbidden induced structures, such as triangle-free graphs.
- Proving the 1-2-3 Conjecture for claw-free graphs and  $2K_2$ -free graphs.

# Perspectives and conclusion

A few perspectives :

- Proving the  $(2,2)$ -Conjecture for other graph classes with forbidden induced structures, such as triangle-free graphs.
- Proving the 1-2-3 Conjecture for claw-free graphs and  $2K_2$ -free graphs.

Thank you for your attention !