Strongly locally irregular graphs and decompositions

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Irregularity

Irregularity

What is an irregular graph?

Irregularity

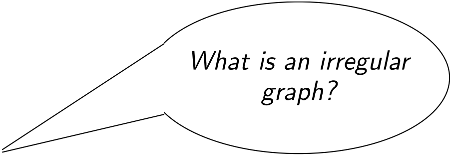
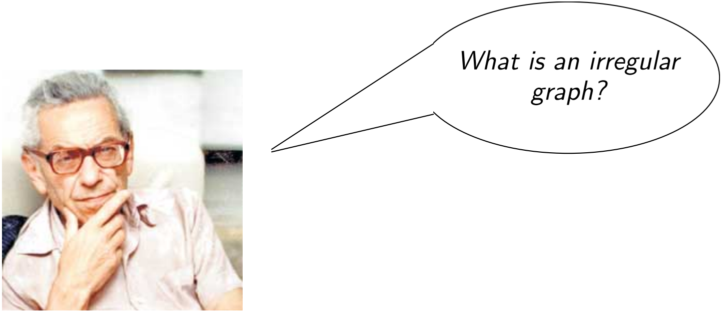


Figure: Some guy thinking [Chartrand, Erdös and Oellermann 88].

Irregularity



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Observation

There is no graph other than K_1 where every two vertices have different degree.

Local irregularity

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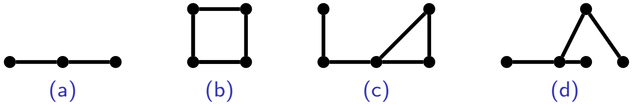

Figure: Locally irregular (or not) graphs

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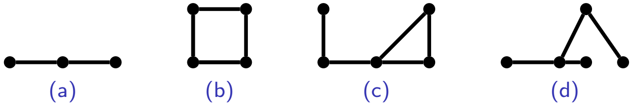

Figure: Locally irregular (or not) graphs

Observation

Not all graphs are locally irregular.

L.i. decompositions

Definition

A *decomposition* of a graph G is a partition of the edges of G. A *locally irregular decomposition* (l.i. decomposition) is a decomposition in locally irregular graphs.

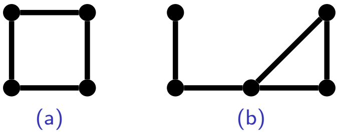

Figure: Two graphs that are not locally irregular

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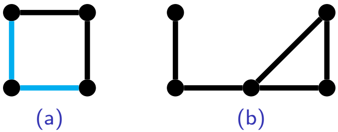

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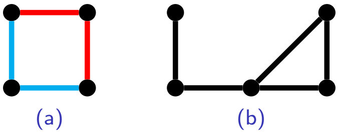

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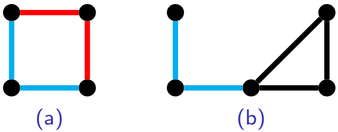

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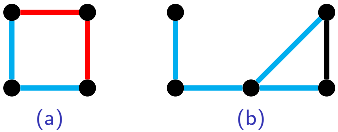

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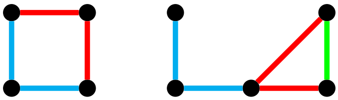

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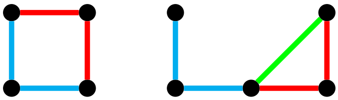

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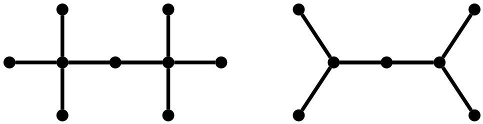

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Figure: A strongly locally irregular graph and one that is no s.li.

Decomposability

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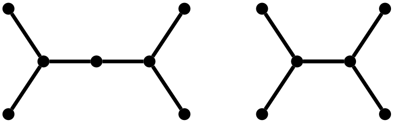

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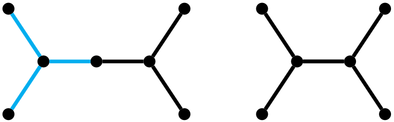

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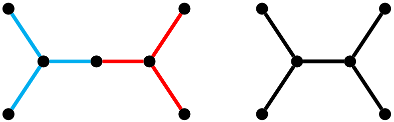

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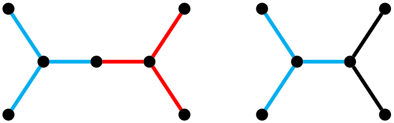

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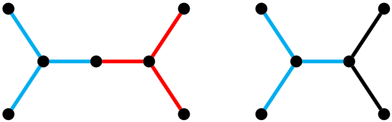

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Theorem

Given a graph G, deciding if G admits a s.l.i. decomposition is NP-hard.

This stays true for planar graphs of maximum degree 4.

Some decomposable classes

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If G is a graph with minimum degree at least 6, then G admits a s.l.i. decomposition.

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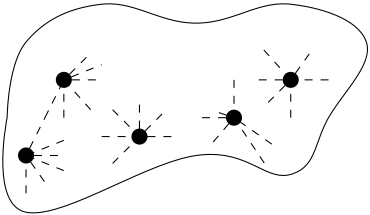

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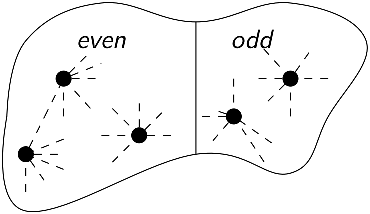

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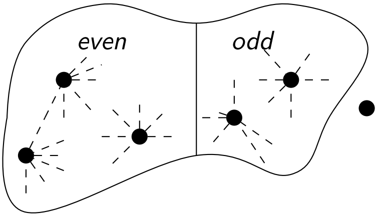

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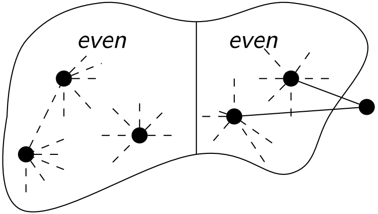

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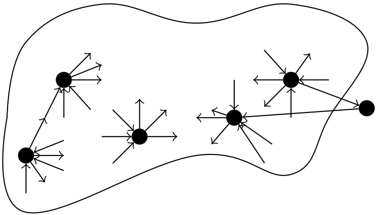

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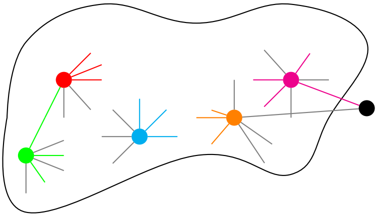

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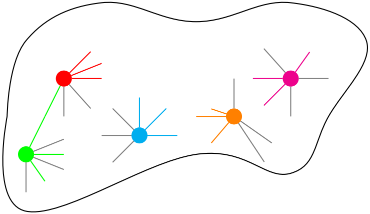

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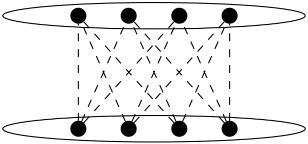

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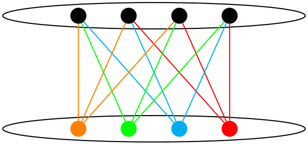

Subcubic graphs

Theorem

A subcubic graph G is s.l.i decomposable if and only if there exists a bipartition of G where all vertices of degree less than 2 are in the same part.

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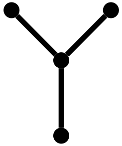

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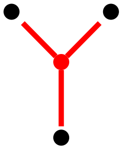

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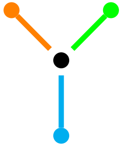

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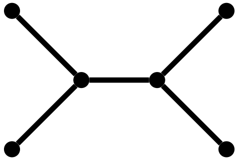

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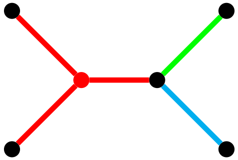

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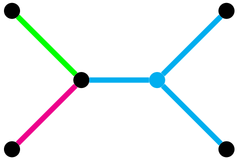

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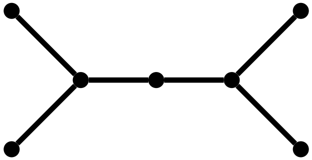

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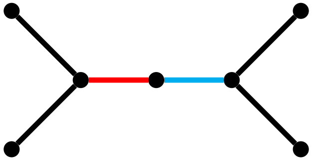

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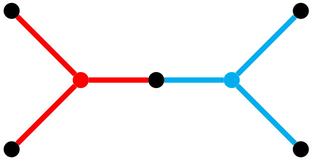

The $\chi_{\mathrm{s.l.i.}}$ parameter

Definition

For a graph G, we denote $\chi_{\mathrm{s.l.i.}}(G)$ the smallest k such that G admits a *s.l.i. k-edge-colouring*.

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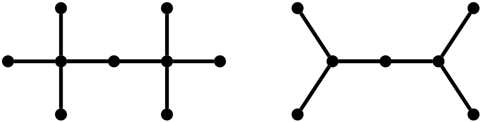

(a) $\chi_{\mathrm{s.l.i.}}(G_a) = 1$ (b) $\chi_{\mathrm{s.l.i.}}(G_b) = 2$

Figure: Graphs with their respective value of $\chi_{\mathrm{s.l.i.}}$.

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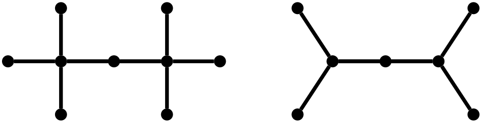

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Figure: Graphs with their respective value of $\chi_{\text{s.l.i.}}$.

Theorem

Given a graph G, deciding if $\chi_{\text{s.l.i.}}(G) \leq 2$ is NP-hard.

This stays true even if G is bipartite.

Graphs with high $\chi_{\mathrm{s.l.i.}}$

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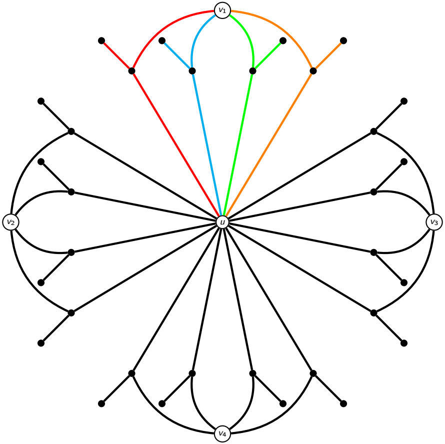

Figure: A graph G with $\chi_{\text{s.l.i.}}(G) = 16$.

$\chi_{\mathrm{s.l.i.}}$ of trees

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In a rooted tree, a *shrub* is the subtree induced by an edge.

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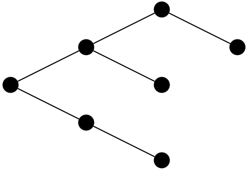

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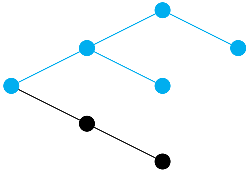

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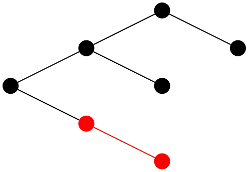

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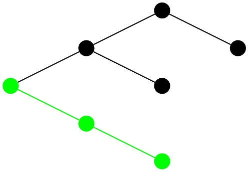
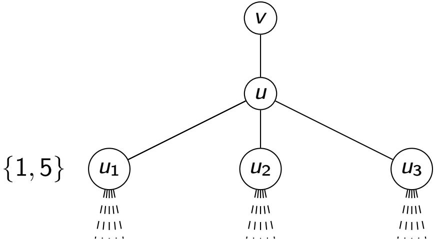


Mending shrub cuttings together

Induction hypothesis

For a shrub uu_i, we can compute all the possible degrees of u_i in the color of uu_i in some *loosely s.l.i. decomposition*.



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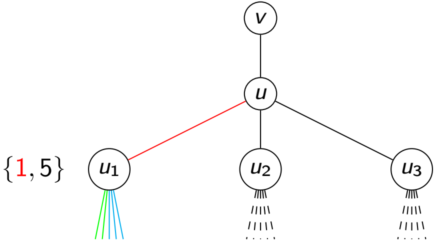

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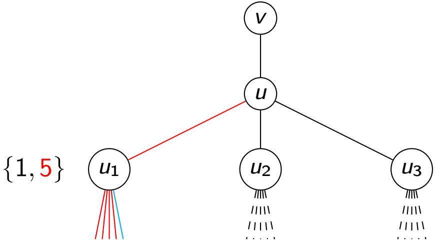

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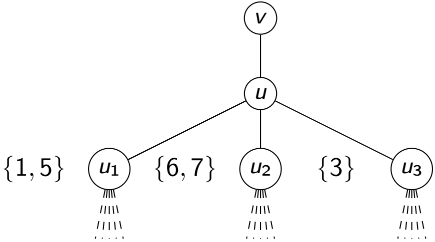

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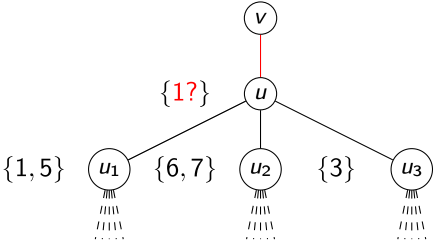

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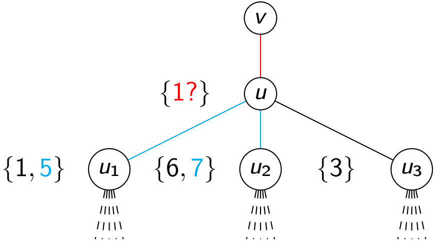

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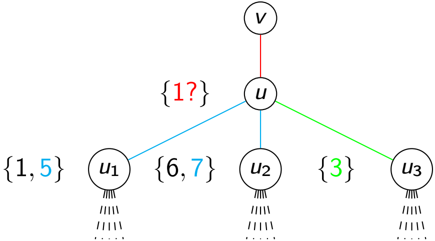

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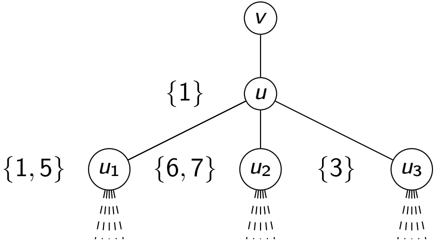

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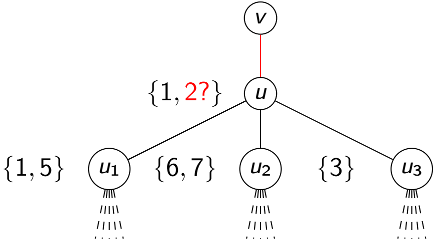

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Thank you!