Monitoring edge-geodetic sets

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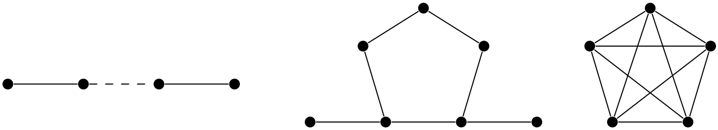
Monitoring edge-geodetics

Definition

A monitoring edge-geodetic set, or MEG-set, of a graph G is a vertex subset $M \subseteq V(G)$ such that given any edge e of G, e lies on every shortest u-v path of G, for some $u, v \in M$. For a graph G, we denote $meg(G)$ the size of a smallest MEG-set of G.



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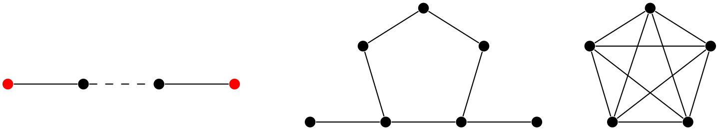

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This object simulates probes monitoring a network: if the value of the ping between two probes increases, then one can know where the failure happened.

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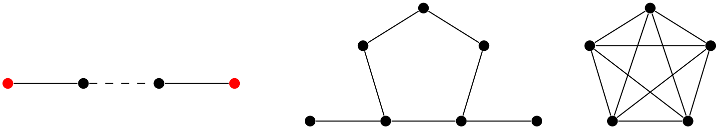

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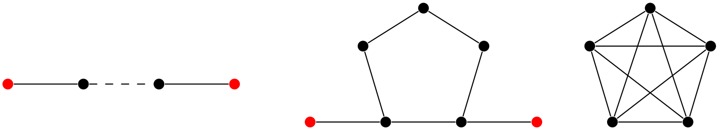

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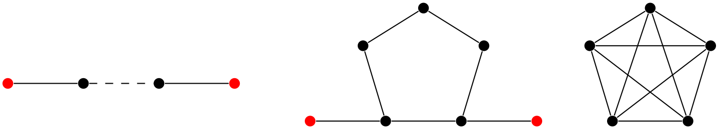

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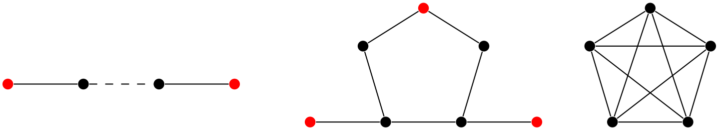

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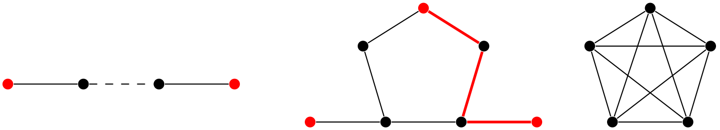

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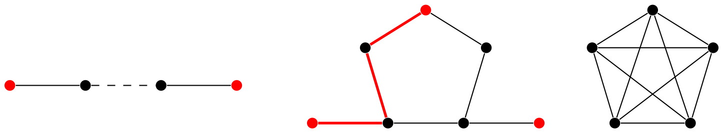

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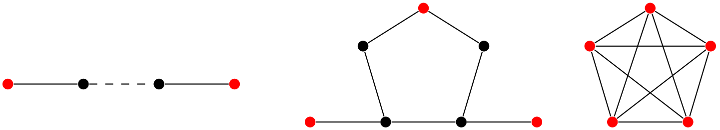

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A few results on the *meg* parameter

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Theorem [Haslegrave, 2023]

The decision problem of determining for a graph G and a natural number k whether $meg(G) \leq k$ is NP-complete.

Preliminary observations

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We can consider 2-connected graphs.

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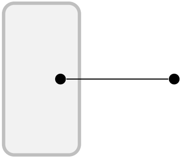

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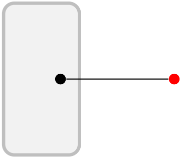

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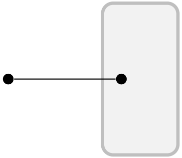

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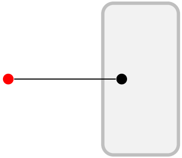

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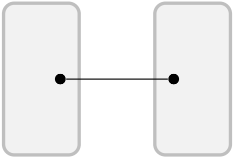

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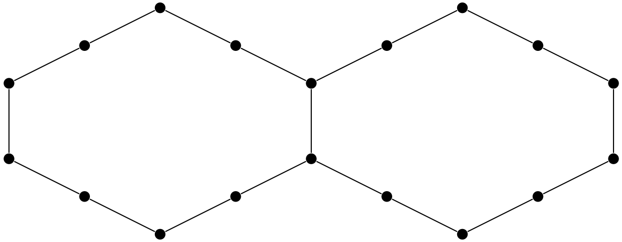
Relation between meg and girth

Theorem [Foucaud *et al.* (2023+)]

Let G be a graph. If G has n vertices, and girth g, then $meg(G) \leq \frac{4n}{g+1}$.



We use a simple algorithm.

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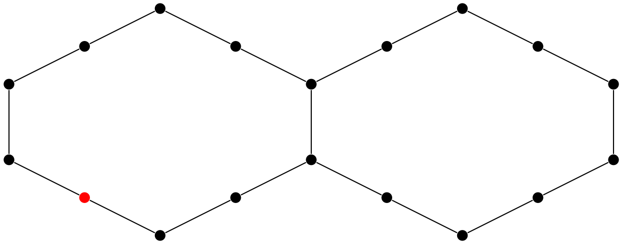

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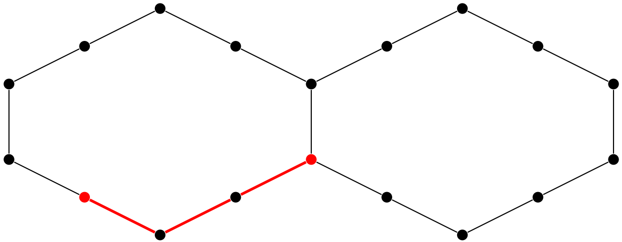

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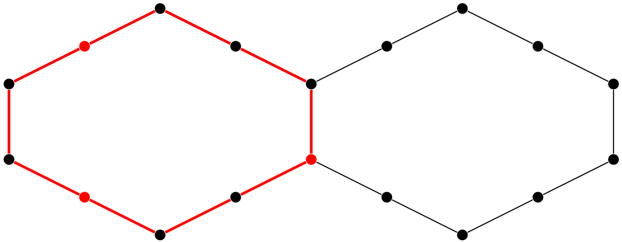

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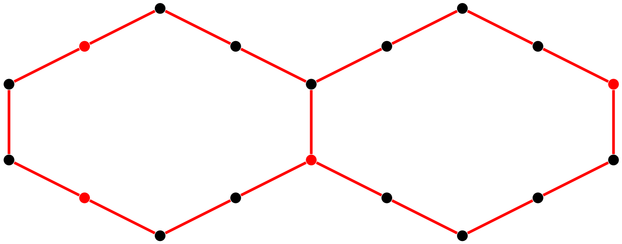

And it forms a MEG-set.

Relation between meg and girth

Claim

Such a set is a MEG-set.

Oriented version

We consider orientations of simple graphs, without digones.

Definition

In an oriented graph $\overrightarrow{G}$, two vertices x and y are said to monitor an arc $\overrightarrow{a}$ if $\overrightarrow{a}$ belongs to all oriented shortest paths from x to y or from y to x.

Definition

A monitoring arc-geodetic set, or MAG-set, of an oriented graph $\overrightarrow{G}$ is a vertex subset $M \subseteq V(\overrightarrow{G})$ such that given any arc $\overrightarrow{a}$ of $A(\overrightarrow{G})$, $\overrightarrow{a}$ is monitored by x, y, for some $x, y \in M$. For an oriented graph $\overrightarrow{G}$, we denote $mag(\overrightarrow{G})$ the size of a smallest MAG-set of $\overrightarrow{G}$.

First results

First note that for an oriented graph $\overrightarrow{G}$, the relation between $mag(\overrightarrow{G})$ and $meg(G)$ is not clear:

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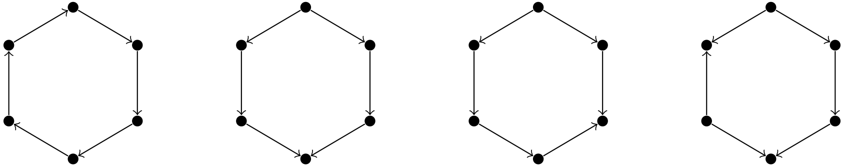
Theorem [Das *et al.*, 2023+]

Let $\overrightarrow{G}$ be an oriented tree. There is a unique minimal MAG-set to $\overrightarrow{G}$, and it is exactly the set of sources and sinks of $\overrightarrow{G}$.

Cycles

Theorem [Das *et al.*, 2023+]

The MEG-set of cycles are completely characterized.



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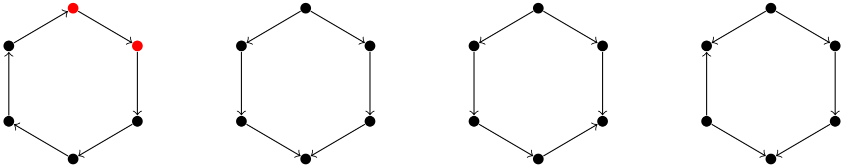

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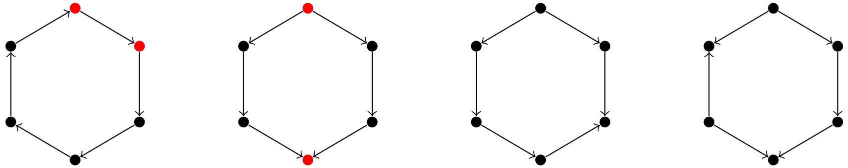

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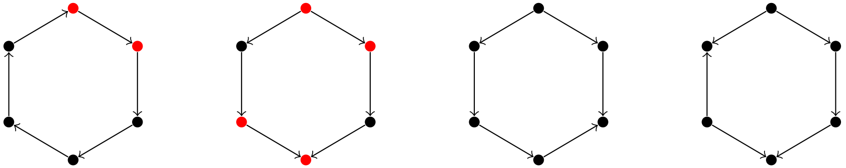

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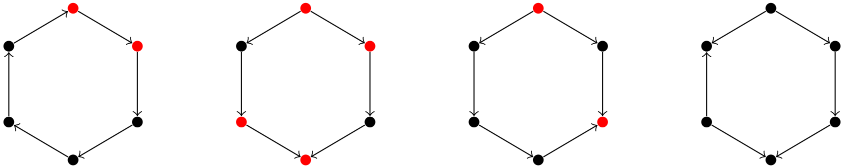

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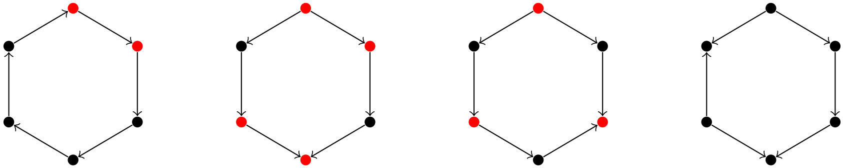

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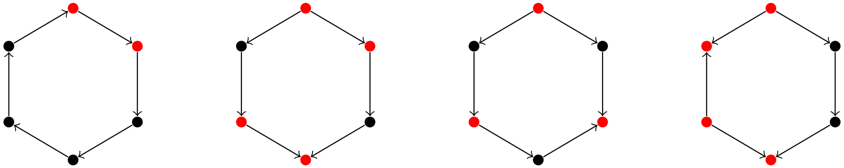
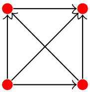


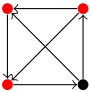
Tournaments

Theorem [Das *et al.*, 2023+]

Let $\overrightarrow{G}$ be an orientation of K_n for some $n \in \mathbb{N}^*$. Then $mag(\overrightarrow{G}) \in \{n-1, n\}$.

Since one can check in polynomial type if a set of vertices of $\overrightarrow{G}$ is an MAG-set, we can now easily characterize all tournaments for this parameter.





Complexity of computing the MAG-set size

We consider the following decision problem:

MAG-SET problem
Instance: An oriented graph $\overrightarrow{G}$, an integer k.
Question: Does there exist an MAG-set of $\overrightarrow{G}$ of size k?

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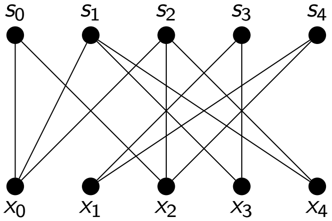
Theorem [Das *et al.*, 2023+]

The MAG-SET problem is NP-complete.

The SETCOVER problem

We proceed with a reduction from the SETCOVER problem.



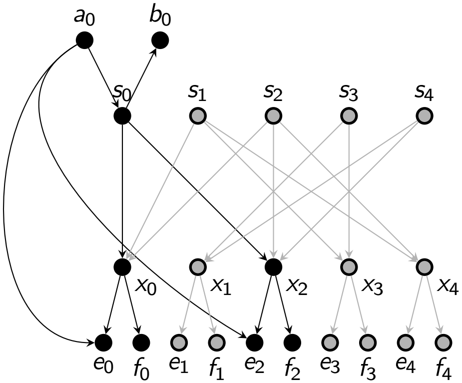
SETCOVER Problem:

Instance: A set $\{X_0, X_2, \cdots, X_n\}$, sets $\{S_0, S_1, \cdots, S_m\}$ such that $\cup_{i=0}^{m} S_i = \{X_0, X_2, \cdots, X_n\}$ and an integer k.

Question: Does there exist a subcollection of at most k sets S_i's such that their union is $\{X_0, X_2, \cdots, X_n\}$.

Our gadget for the reduction

From any instance of SETCOVER I, we can compute $\overrightarrow{G(I)}$ an instance of MAG-SET. Assume we have M an MAG-set of $\overrightarrow{G(I)}$.



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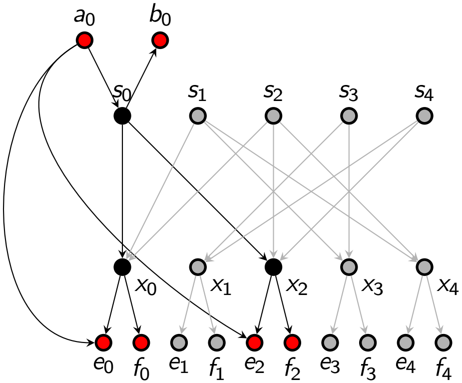

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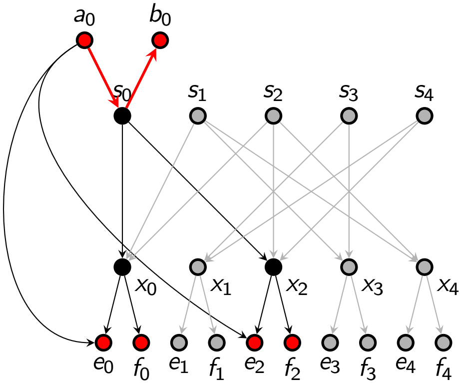

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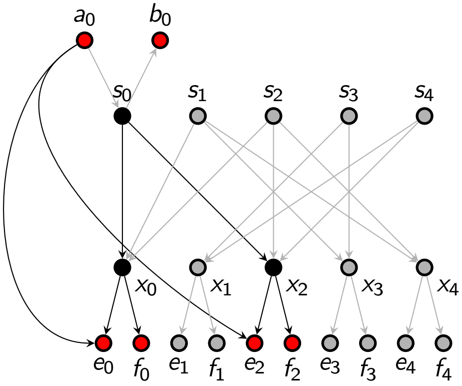

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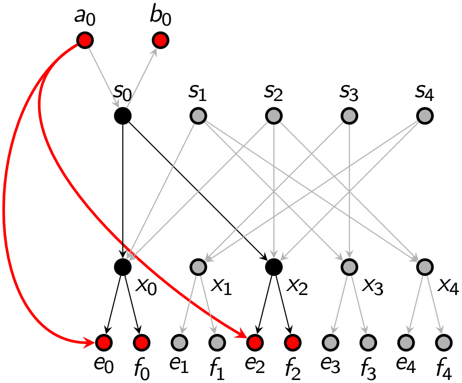

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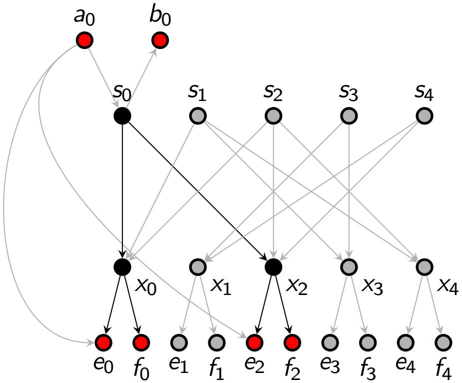

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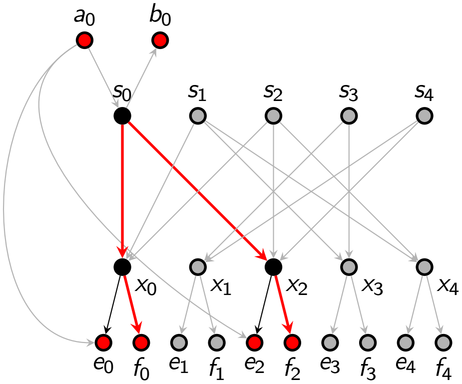

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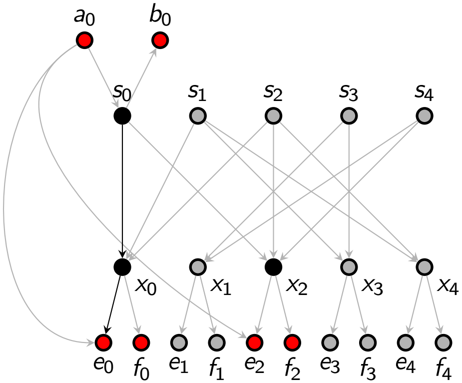

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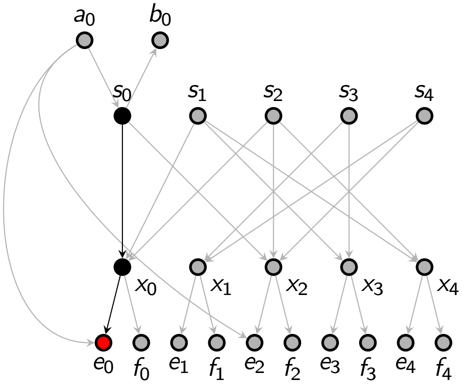

We now need to study the properties of any MAG-set of $\overrightarrow{G(I)}$.

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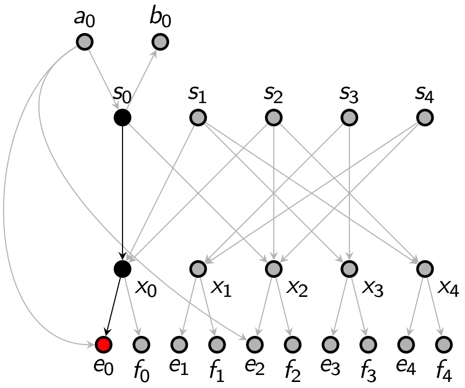

For every X_i, either x_i or some s_j with $X_i \in S_j$ is in M.

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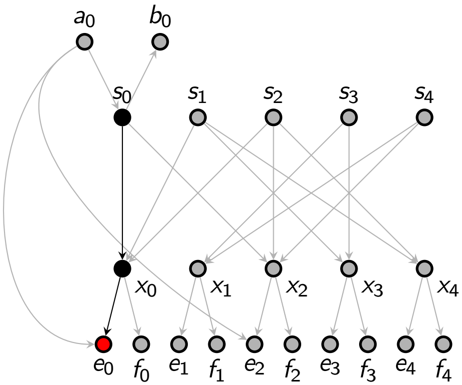

If $s_j \in M$ then we are done !

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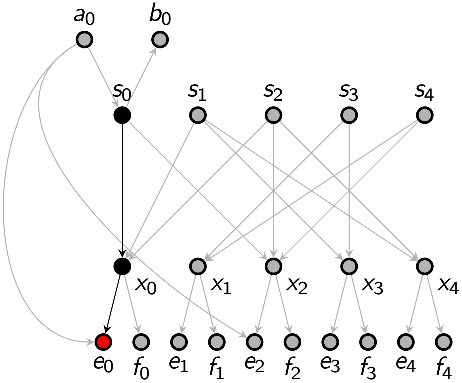

If some $x_i \in M$, then we remove it and add an arbitrary s_j to M, with $X_i \in S_j$.

Conclusion

We have proven the following results on oriented graphs:

	non-oriented	oriented
Trees	leaves	sources and sinks
Cycles	3(4 for C_4)	$2 \leq mag \leq n$
K_n	n	either $n-1$ or n
Decision problem	NP-hard	NP-hard

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A few perspectives:

- To follow up on the idea of networks, we can study the interaction of monitoring with local constraints on all subgraphs.
- Some other results have been proven for the non-oriented case and the bounds are not known in the oriented case.
- We proved that MAGSET is hard on DAG. One can also wonder if the MAGSET problem is still hard for simpler graph structures, like planar graphs.

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Thank you for your attention!