Distinction and Detection Problems in Graphs

Clara Marcille

Jury:

Éric Duchêne (Université Lyon 1)........................reviewer
Nicolas Nisse (Inria d'Université Côte d'Azur).........reviewer
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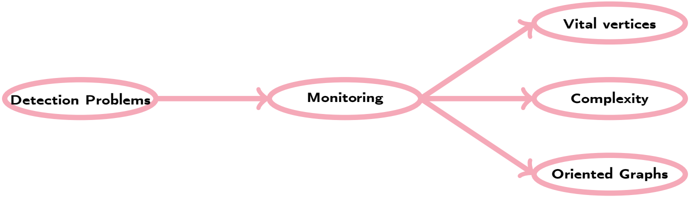
Aline Parreau (Université Lyon 1)........................examiner
Cléophée Robin (Université Paris Cité)..................examiner
Éric Sopena (Université de Bordeaux)....................examiner
Julien Bensmail (Université Côte d'Azur).............supervisor
Hervé Hocquard (Université de Bordeaux)............supervisor

PhD. Defence, 24 June 2025

Detection Problems in Graphs



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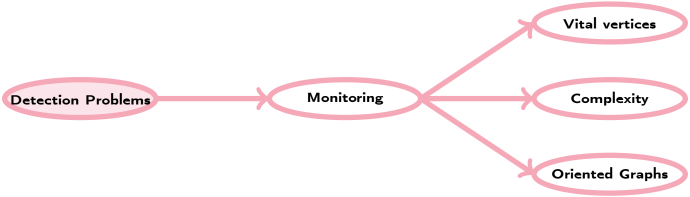

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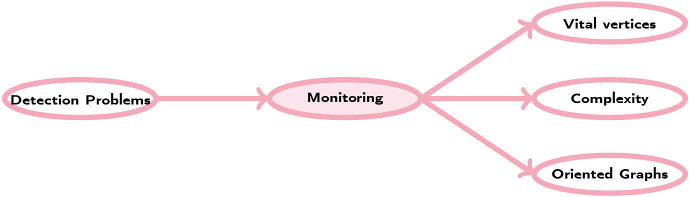

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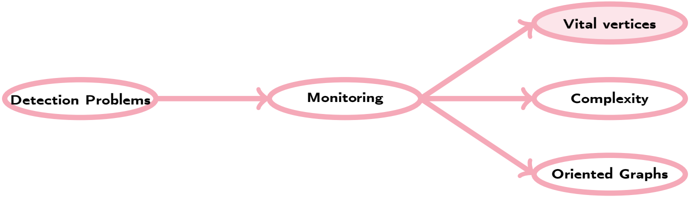

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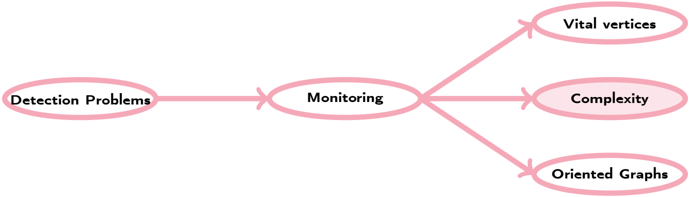

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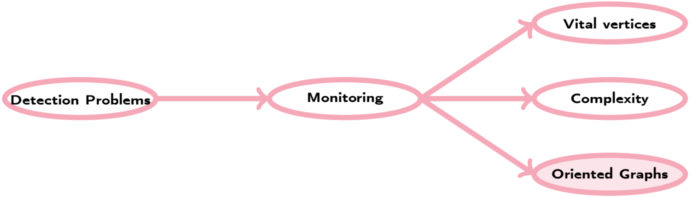

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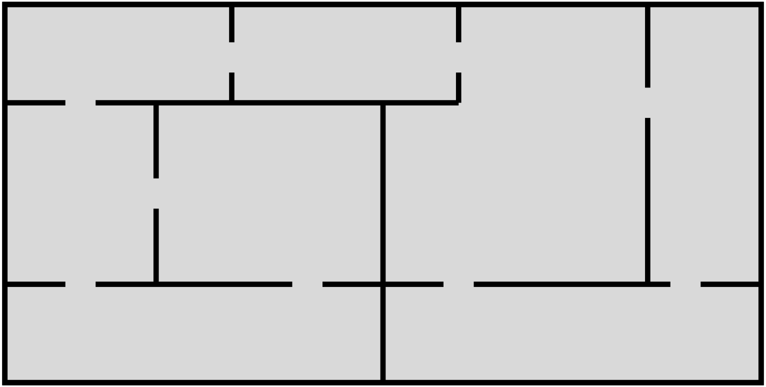

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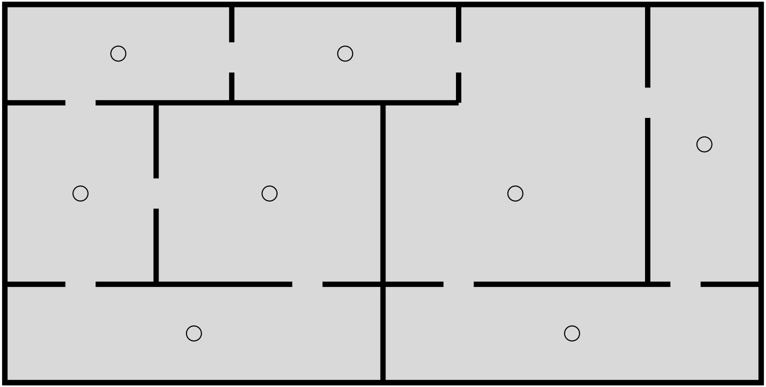

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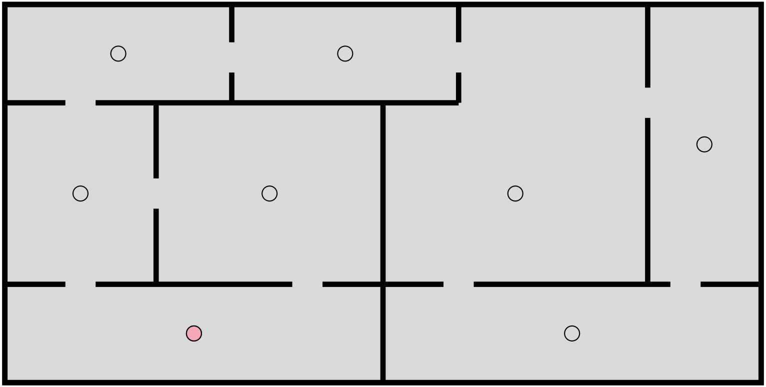

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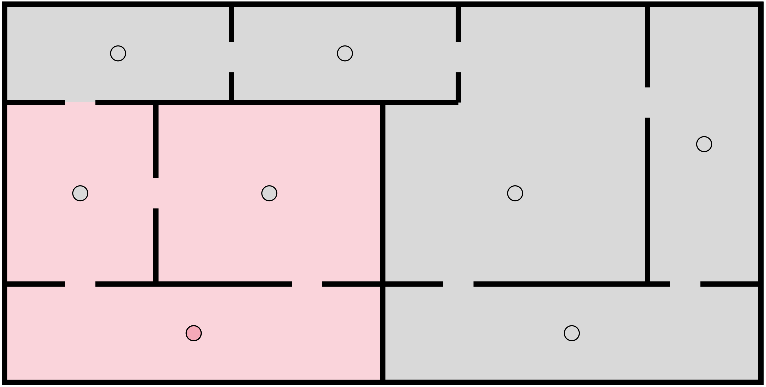

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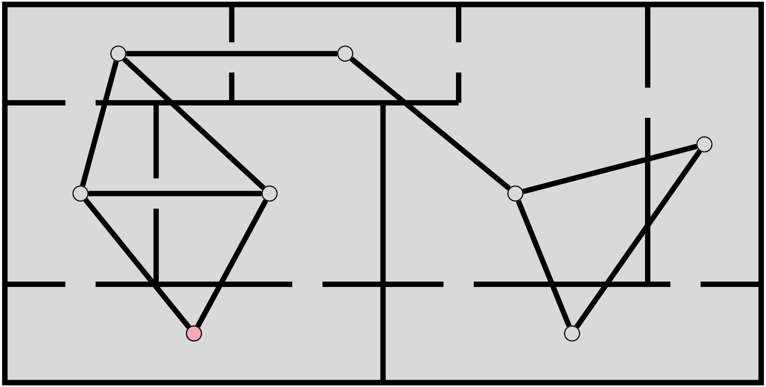

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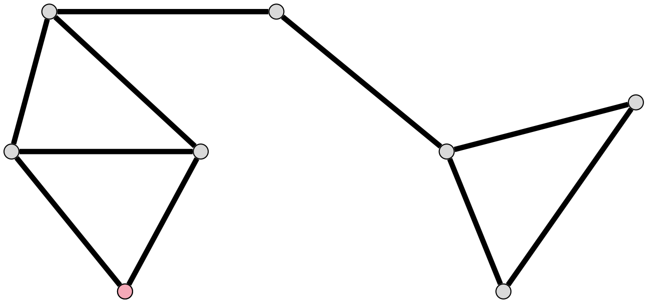

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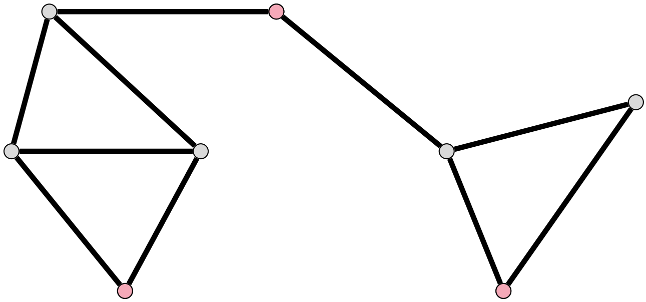
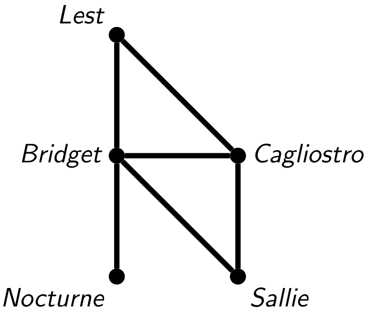


Monitoring Networks through Distances



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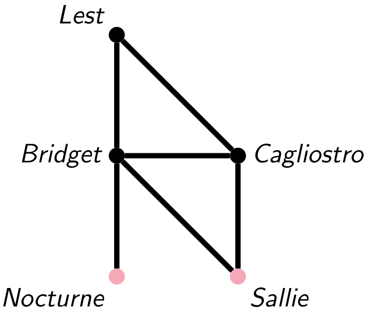

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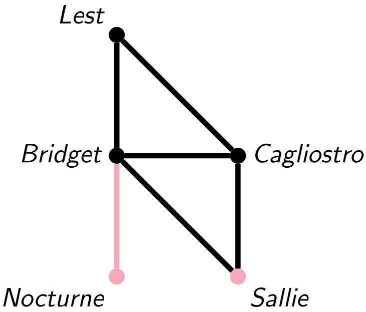

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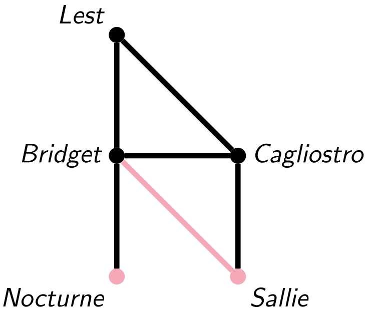

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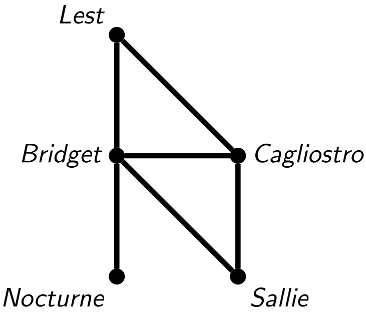

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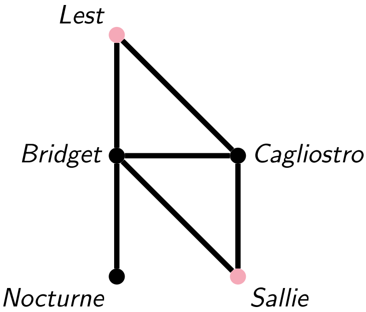

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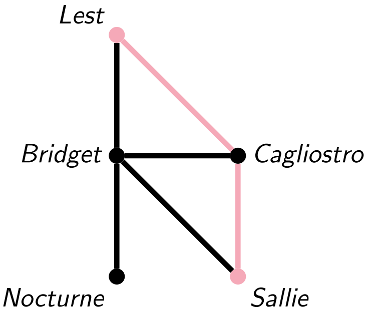

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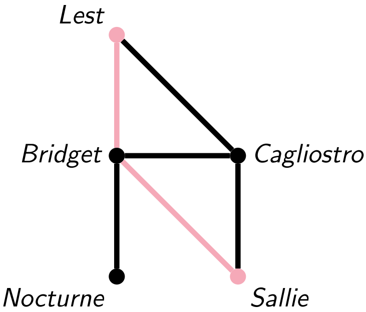

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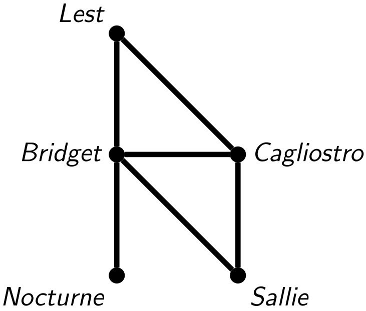

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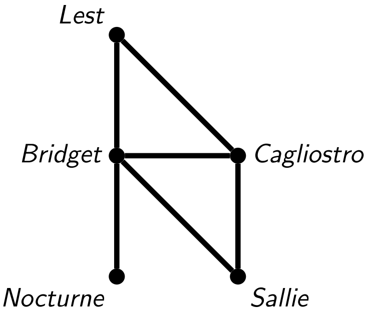

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- messages have a timestamp;
- every step takes the same time;
- messages take a shortest path;
- everyone knows their distance to others.

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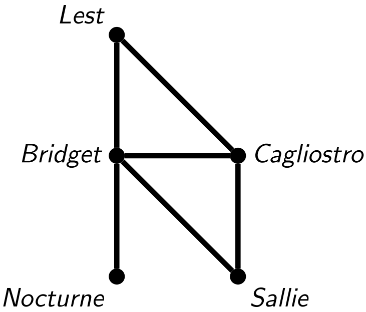

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This object simulates probes monitoring a network: if the value of the ping between two probes increases, then one can know a failure happened.

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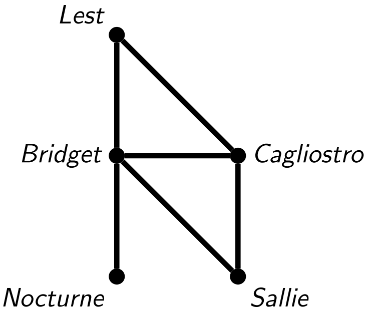

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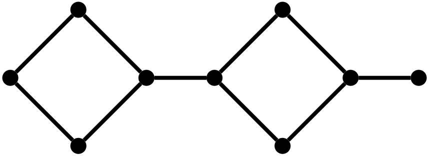
Goal

We want to minimise the number of probes.

Monitoring Edge Geodetics



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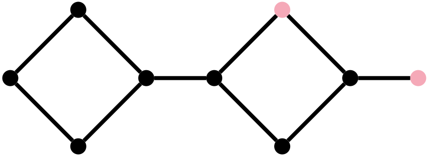

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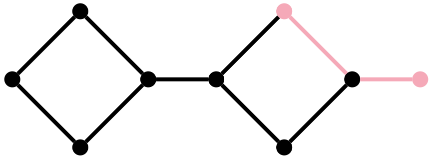

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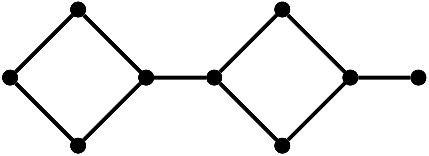

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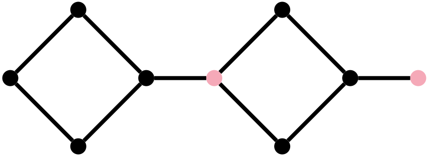

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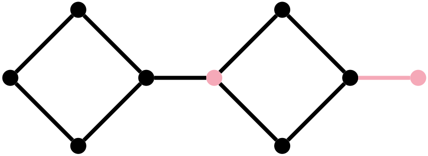

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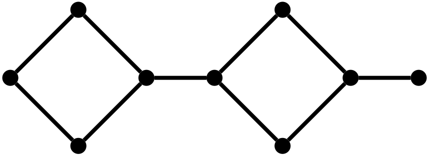

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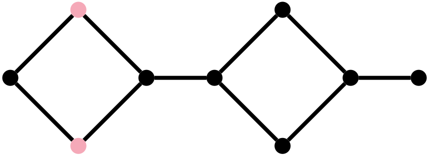

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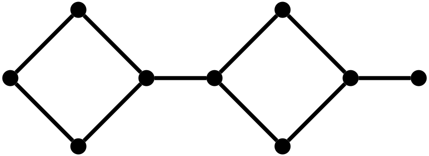

Definition (MEG-set) [FNRS23]

A set M of vertices *monitors* an edge e if e lies on all shortest paths between two vertices of M.

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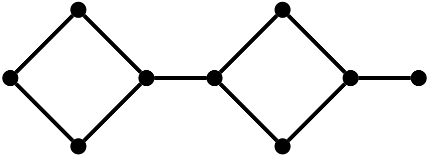

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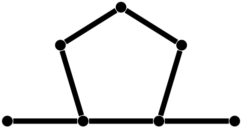

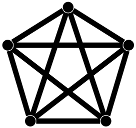

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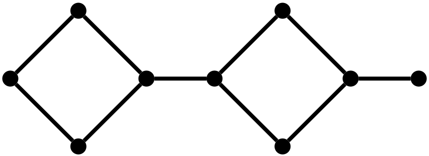

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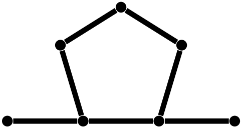

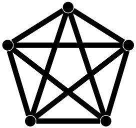

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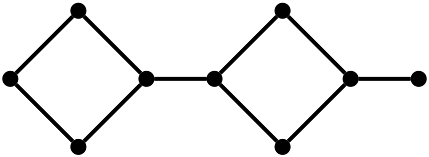

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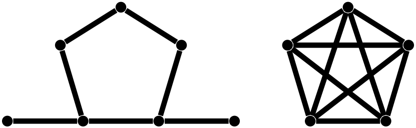

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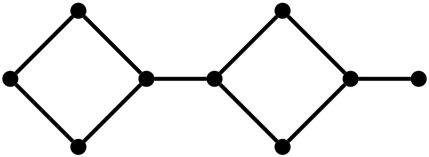

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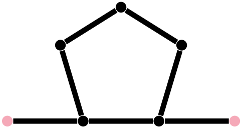

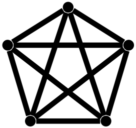

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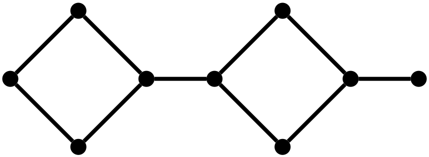

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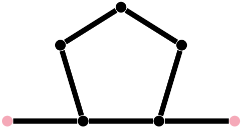

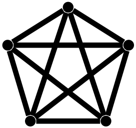

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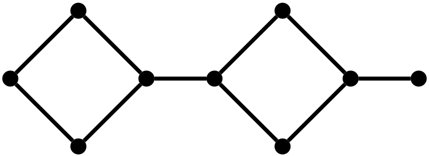

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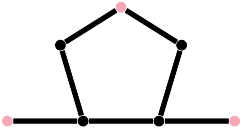

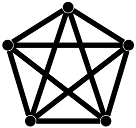

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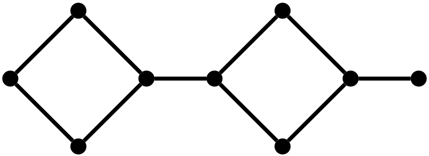

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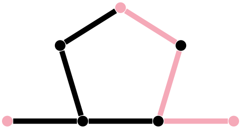

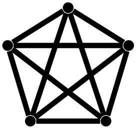

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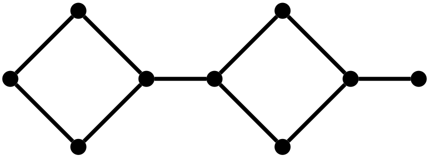

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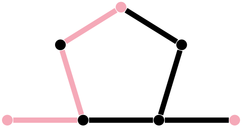

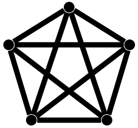

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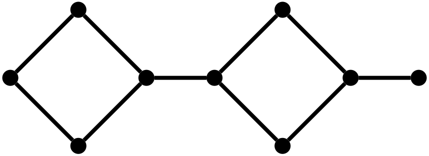

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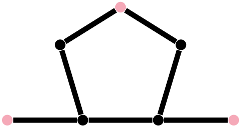

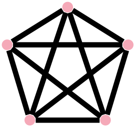

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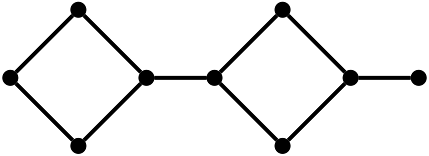

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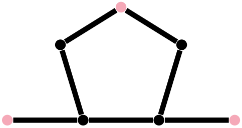

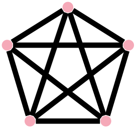

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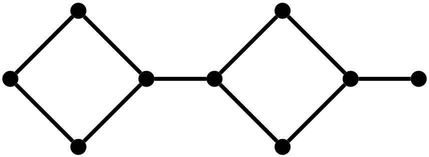

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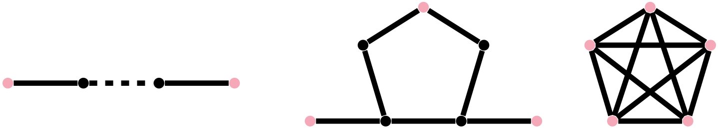


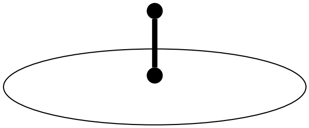
Theorem [Haslegrave, 2023]

Deciding for a graph G and a natural number k whether $meg(G) \leq k$ is NP-complete.

Vital Vertices

Definition

A vertex of G is *vital* if it belongs to all MEG-sets of G.



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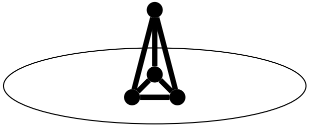

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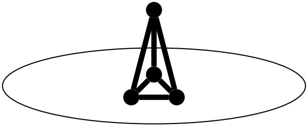

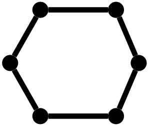

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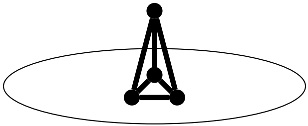

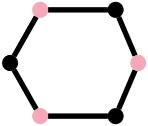

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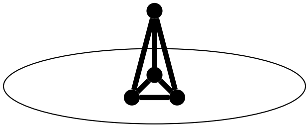

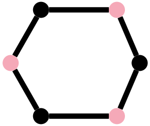

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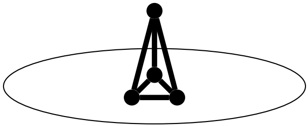

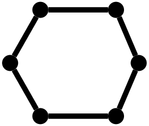

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- block graphs;
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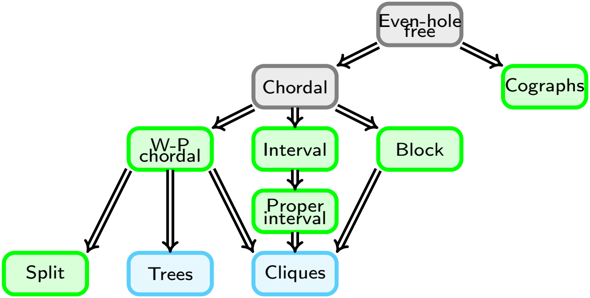

Idea of the Interval Graphs Result

Lemma [FMMSST24]

In an interval graph, a vertex is vital if and only if its neighbourhood has diameter at most 4.

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Parameterised Complexity

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For a graph G, there is a polynomial-time algorithm to:

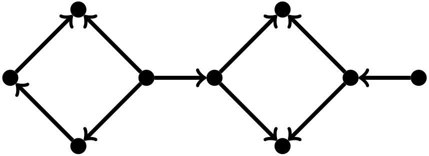
- find a MEG-set of G of size $meg(G) \cdot \sqrt{n \ln m}$, but
- not of size $c \cdot meg(G)$ for any $c > 1$.

Parameterised Complexity

Approximation Algorithms

Oriented version

In fact, we only need to be able to define paths to study monitoring.



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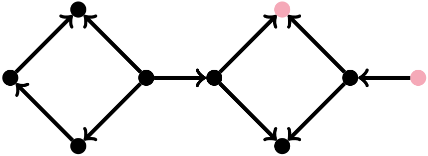

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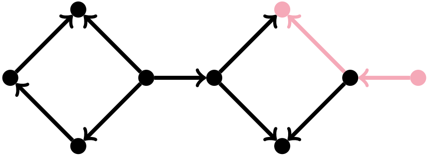

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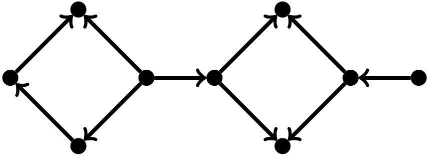

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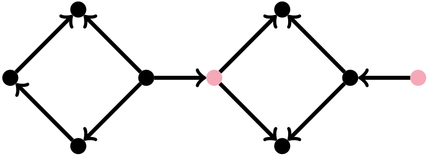

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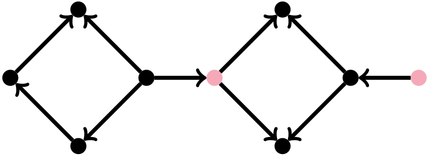

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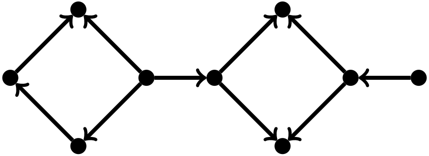

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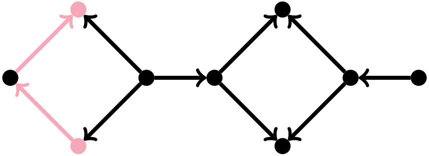

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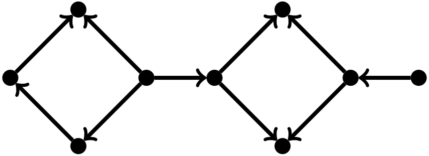

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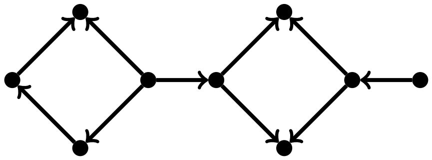

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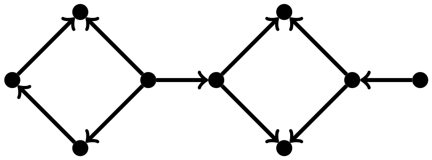

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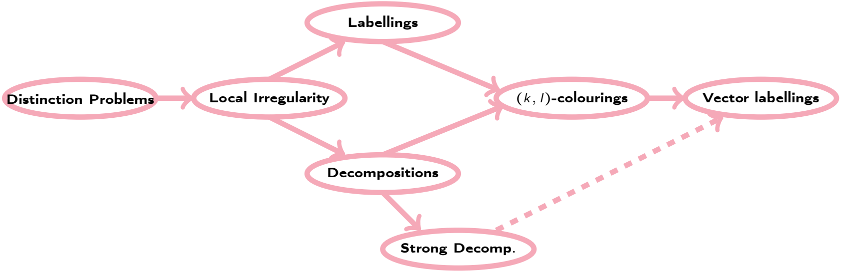

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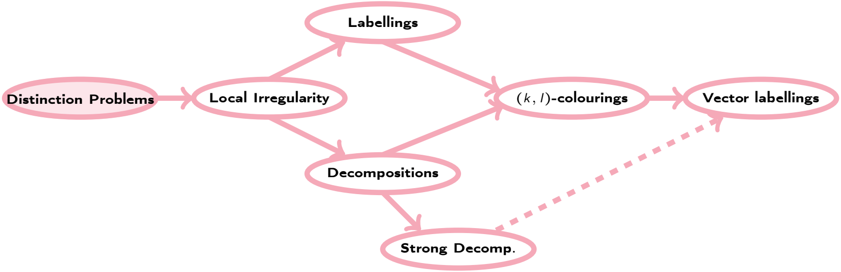

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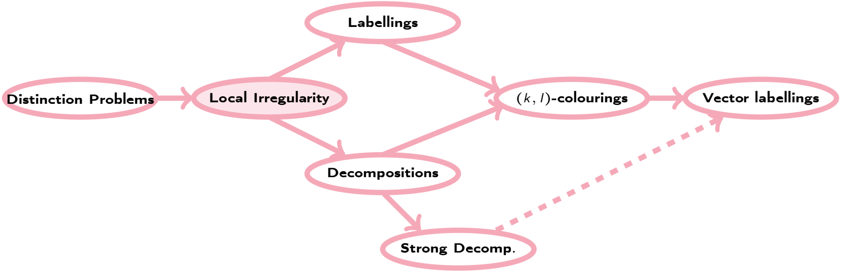

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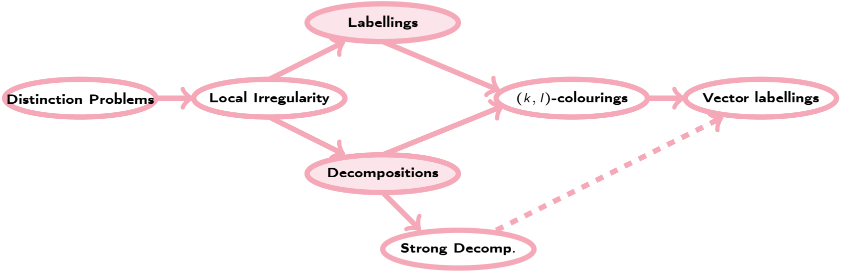

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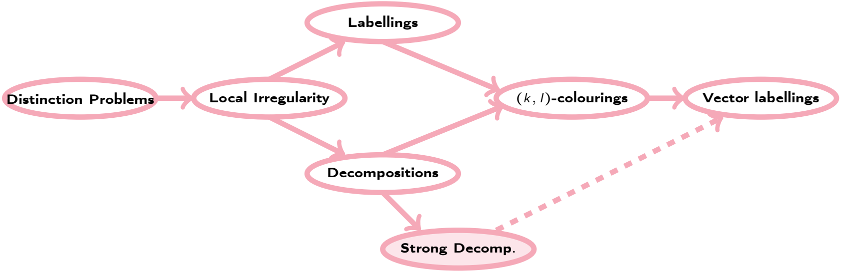

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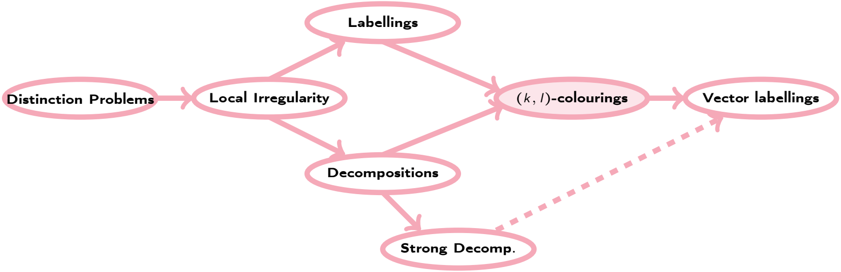

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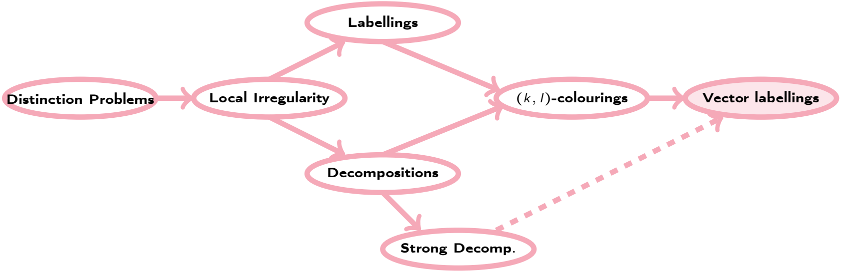

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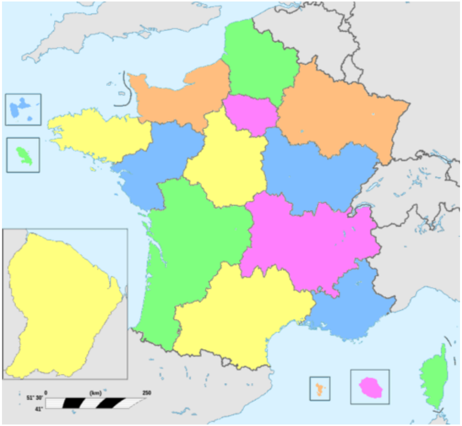

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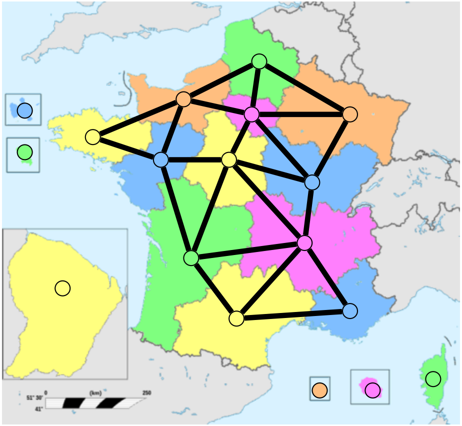

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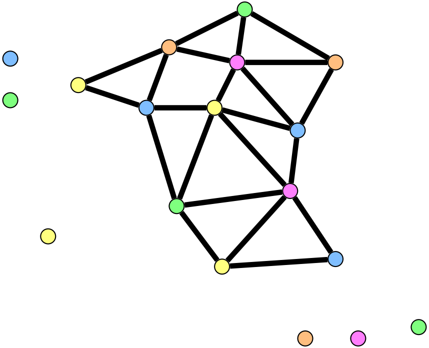

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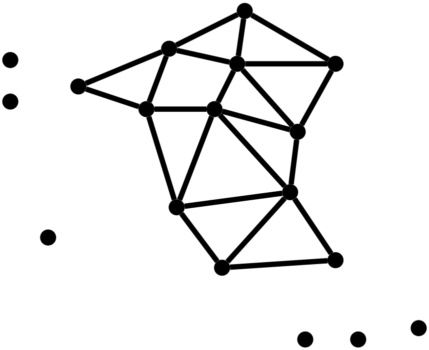

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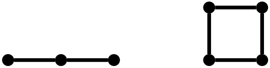

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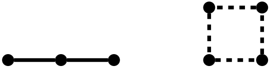

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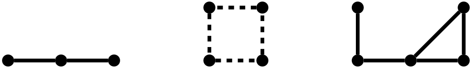

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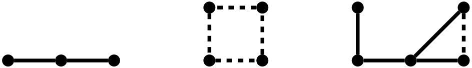

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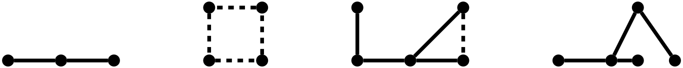

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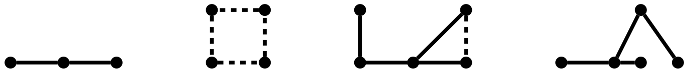
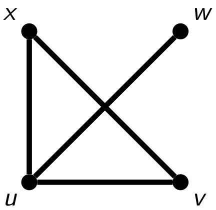
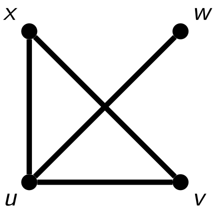


Observation

Not all graphs are locally irregular.

Creating Irregularity

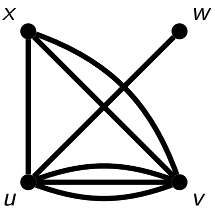


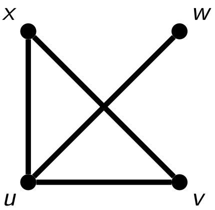


Multigraphs

Decompositions

Creating Irregularity





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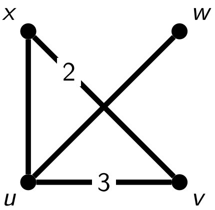

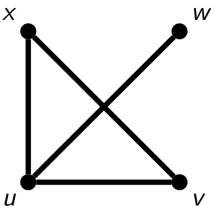

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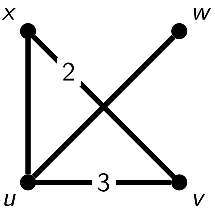

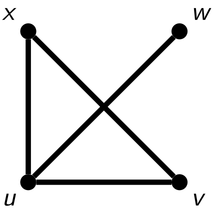

Exceptions	K_2
Lower Bound	3 [DW, 2011]
Upper Bound	3 [Keusch, 2024]

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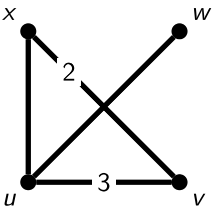

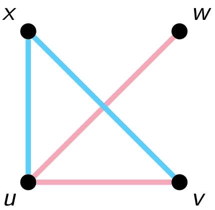

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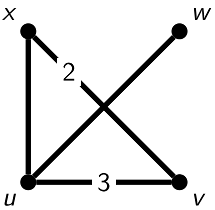

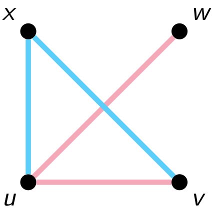

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Exceptions	Polynomial [BB$^+$15]
Lower Bound	4 [SŠ21]
Upper Bound	220 [LPS18]

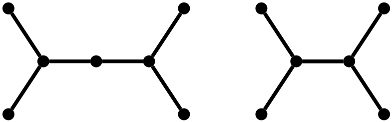
Multigraphs

Decompositions

Strong Local Irregularity

Definition [BM25+]

A graph is *strongly locally irregular* (s.l.i. for short) if the degrees of every two adjacent vertices differ by at least 2.



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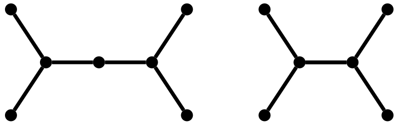

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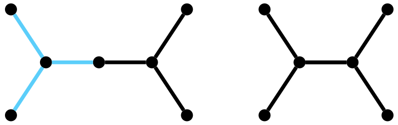

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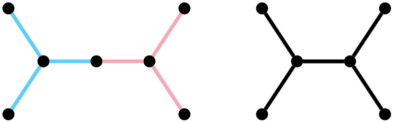

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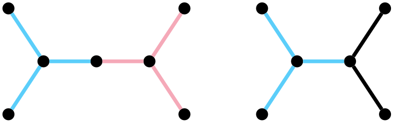

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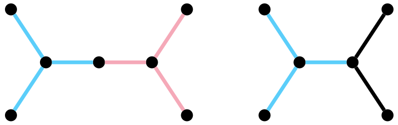

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Class	Decomposability	Lower Bound	Upper Bound
All graphs	NP	16	Unknown
Trees	P	5	16
Subcubic	P	7	7
$\delta \geq 5$	P	Unknown	Unknown

$\chi_{\mathrm{s.l.i.}}$ of Trees

Theorem [BM25+]

If T is a s.l.i. decomposable tree, then $\chi_{\mathrm{s.l.i.}}(T) \leq 16$.

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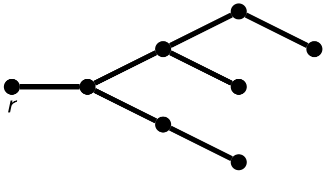

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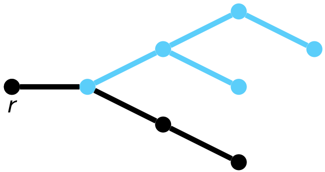

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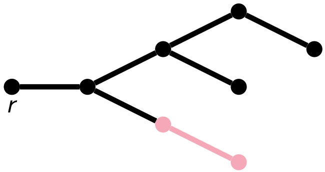

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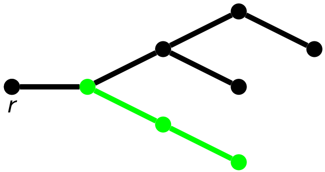

Mending Shrub Cuttings Together

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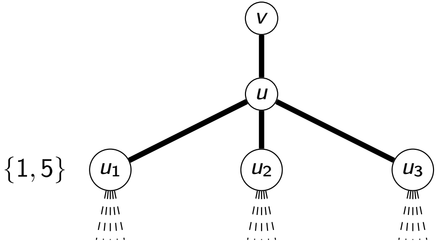

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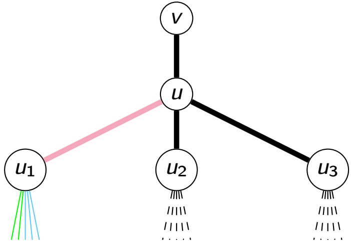

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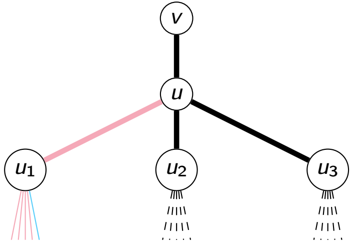

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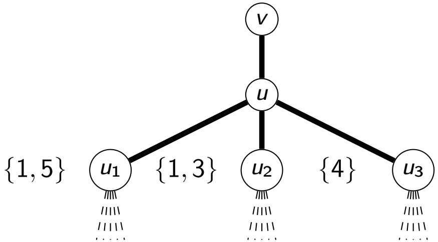

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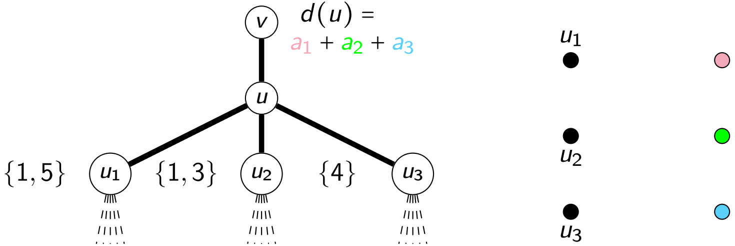

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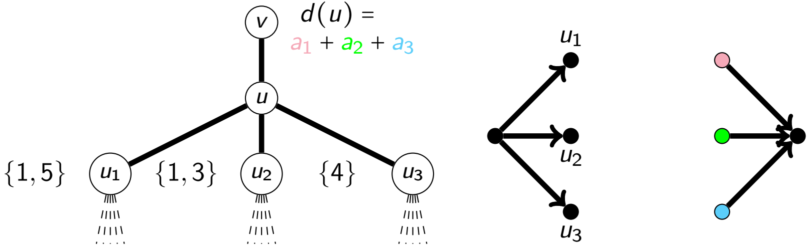

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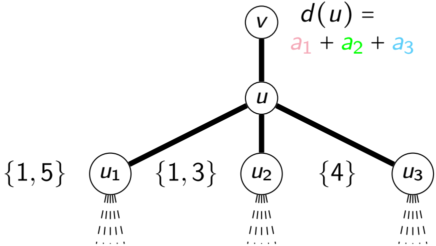

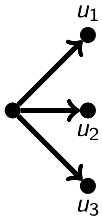

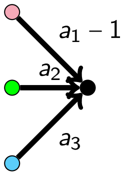

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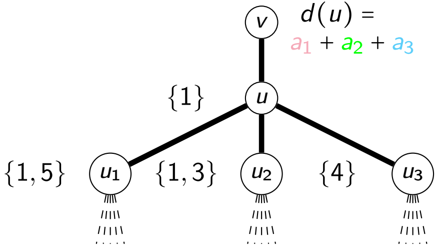

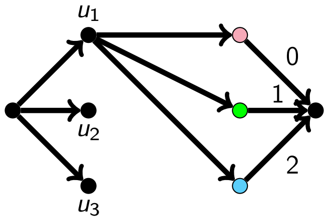

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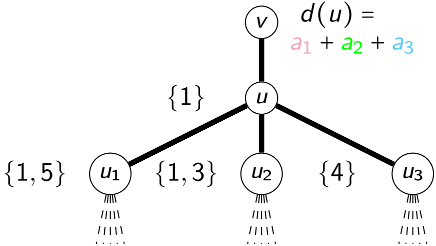

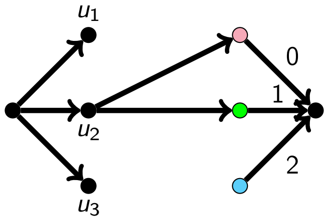

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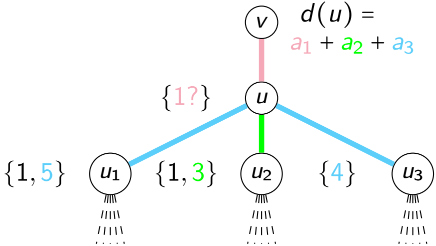

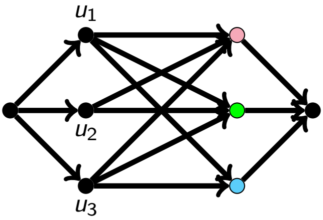

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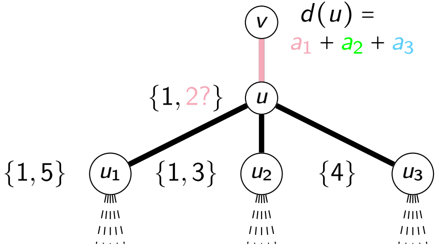

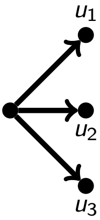

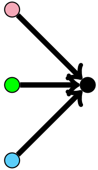

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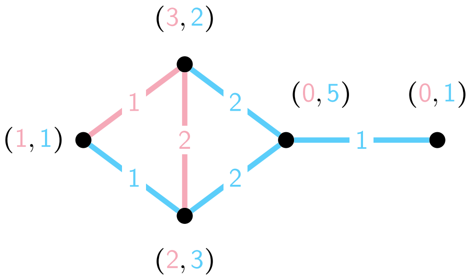
Intuition [BB+19]

If we use both a k-decomposition and a l-labelling, then we may reach better bounds.



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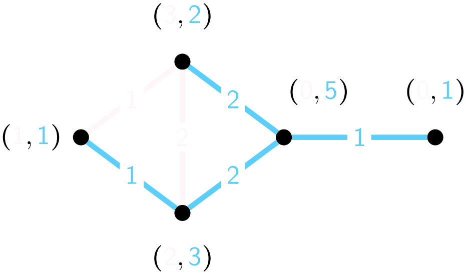

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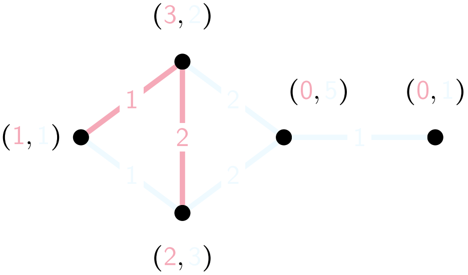

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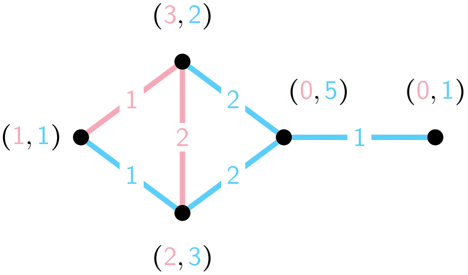

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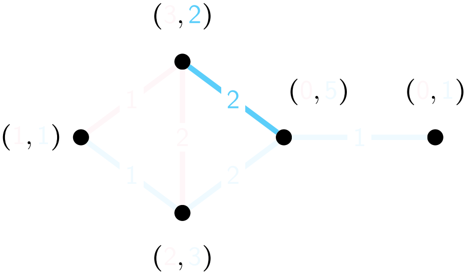

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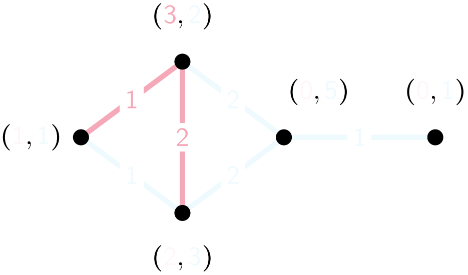

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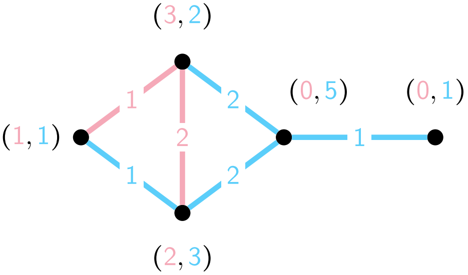

Strong $(2, 2)$ Conjecture [BB+19]

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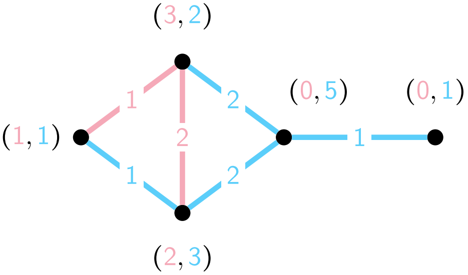

[BHMM25] We prove it for:

- cacti graphs;
- subcubic outerplanar graphs;
- graphs of *maximum average degree* less than $\frac{9}{4}$;
- powers of cycles;
- complete k-partite graphs.

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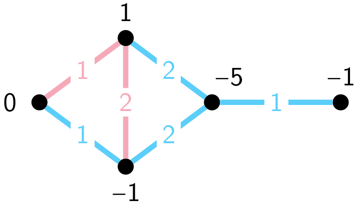

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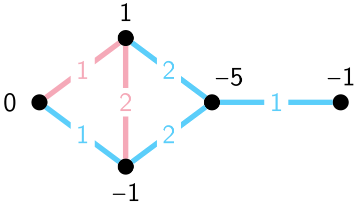

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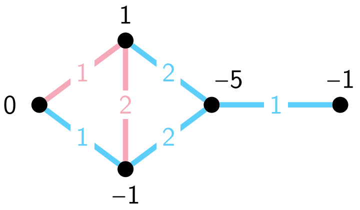

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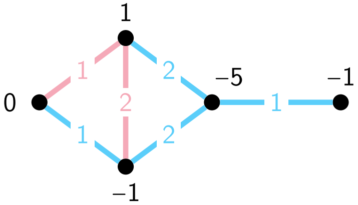

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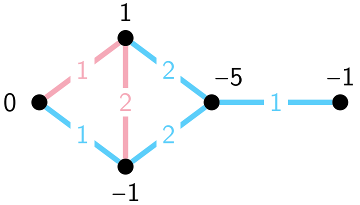

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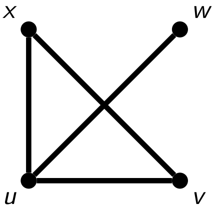
Conjecture (optimal)

← These upper bounds are 2.

(k, l)-colourings in a Euclidean Space

We also introduce a way to encapsulate all previous definitions, as a generalisation of (k, l)-colourings.



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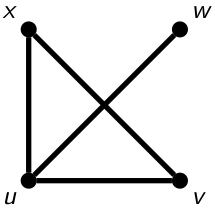

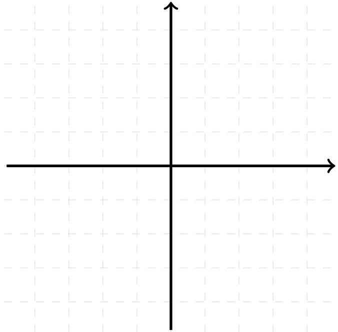

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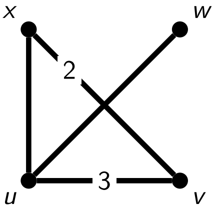

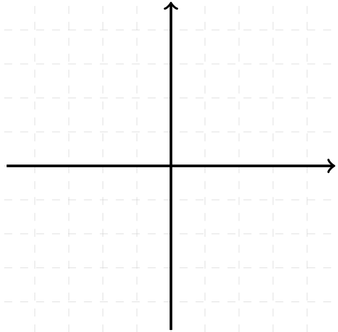

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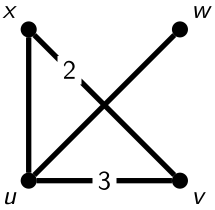

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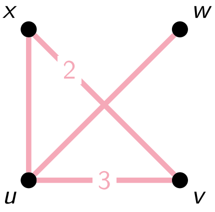

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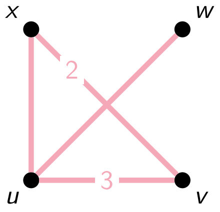

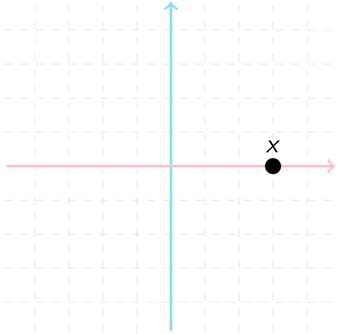

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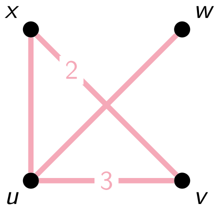

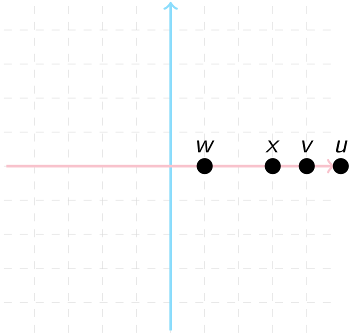

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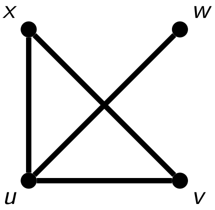

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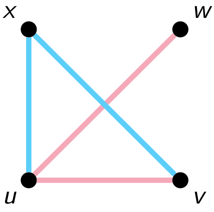

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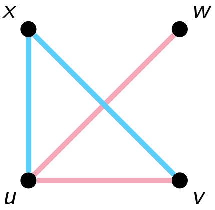

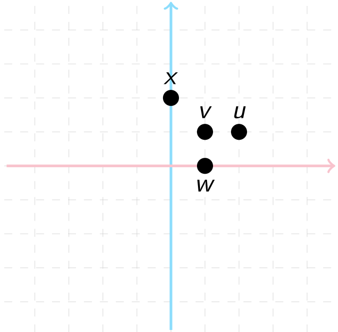

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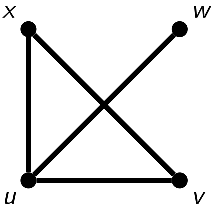

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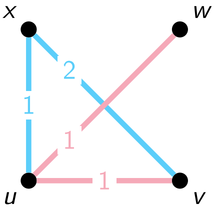

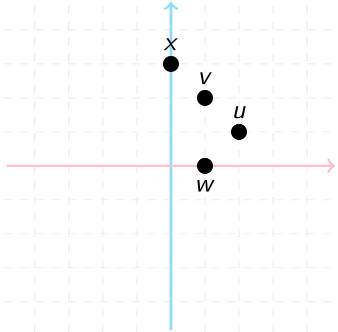

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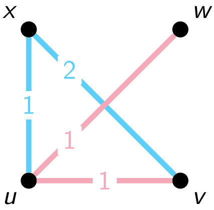

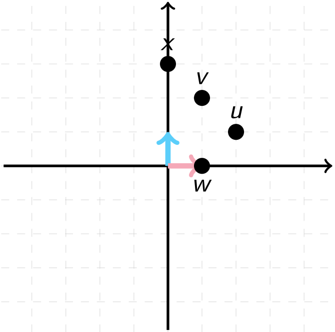

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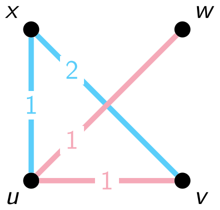

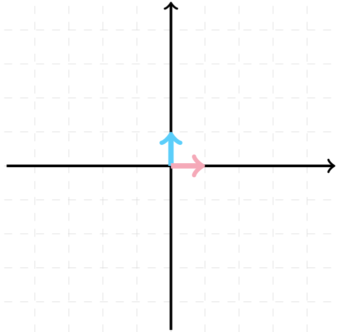

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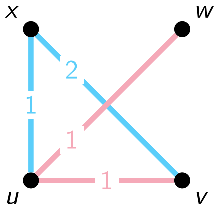

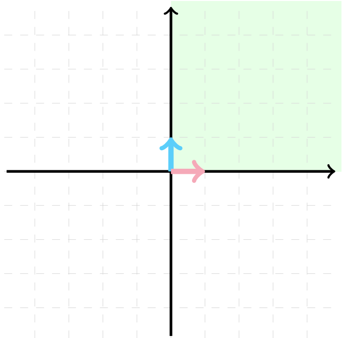

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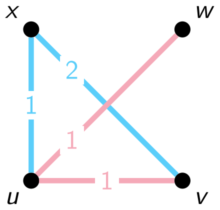

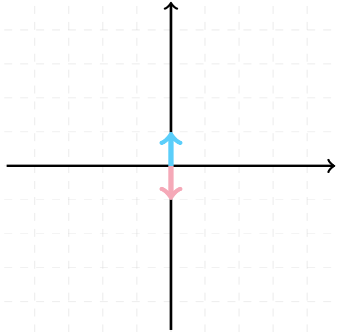

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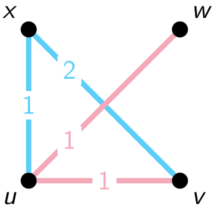

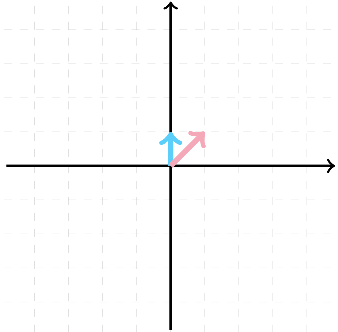

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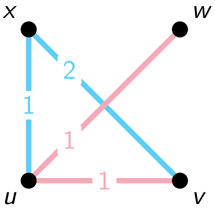

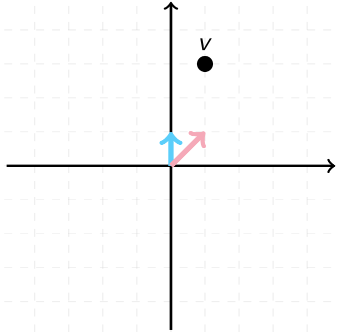

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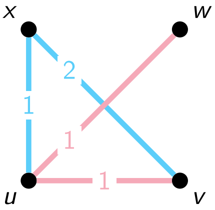

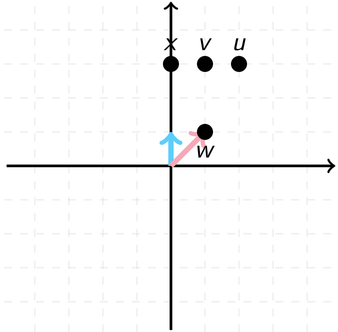

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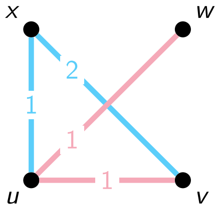

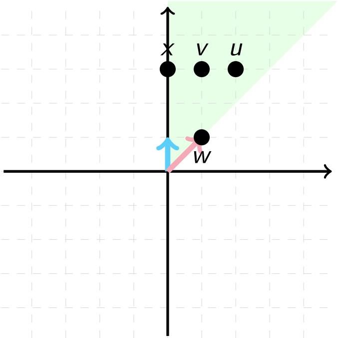

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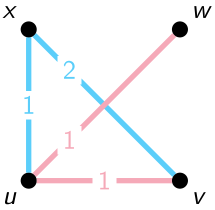

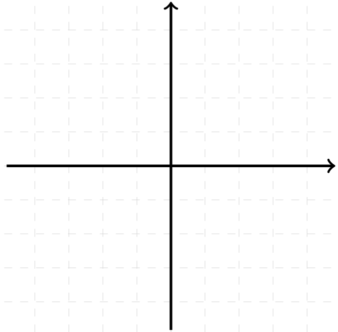

Theorem (informal)

Given a graph G and two vectors, deciding whether G can be properly edge-coloured with these two vectors is NP-hard.

Conclusion on Irregularity

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Conclusion

In this thesis, we tackled multiple problems with:

- structural approaches (studying simple cases, establishing structural properties of solutions), and
- algorithmic approaches (proving hardness, designing parameterised algorithms).

This process can be applied on several variants of a problem, with two goals:

- establishing relations between the variants to prove more results;
- providing a gathering framework to give insights on previous results.

Published Works

Works on Irregularity:

- Going Wide with the 1-2-3 Conjecture, BHM, *DAM*
- On inducing degenerate sums through 2-labellings, BHM, *G&C*
- Adding direction constraints to the 1-2-3 Conjecture, BHM, *TCS*
- On 1-2-3 Conjecture-like problems in 2-edge-coloured graphs, BHMM, *DM*
- Irregularity Notions for Digraphs, BFHM, *G&C*

Journal

- The Weak $(2,2)$-Labelling Problem for Graphs with Forbidden Induced Structures, BHM *CALDAM23*
- An Improved Bound for Equitable Proper Labellings, BM, *IWOCA24*

Conferences

Works on Monitoring:

- Bounds and extremal graphs for monitoring edge-geodetic sets in graphs, FMSST, *DAM*

Journal

- Monitoring Edge-Geodetic Sets in Graphs: Extremal Graphs, Bounds, Complexity, FMMSST, *CALDAM24*

Conferences

What the Future Holds

Submitted works:

- The Strong (2, 2)-Conjecture for more classes of graphs, BBGM;
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- Strongly Locally Irregular Graphs and Decompositions, BM;
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Thank you!

Proof of one of our Results

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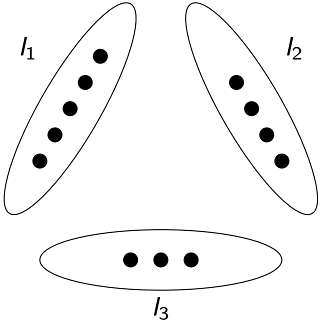
If G is a complete k-partite graph other than K_2 and K_3, then G admits a strong $(2, 2)$-colouring.

Case $k = 3$



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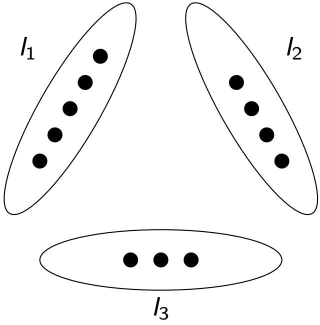

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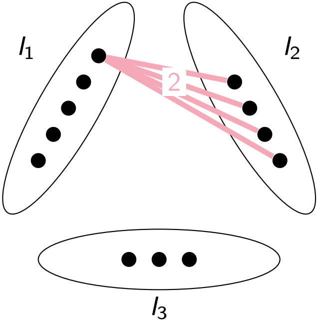

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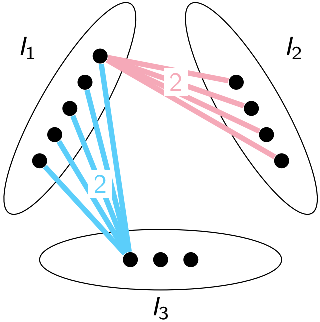

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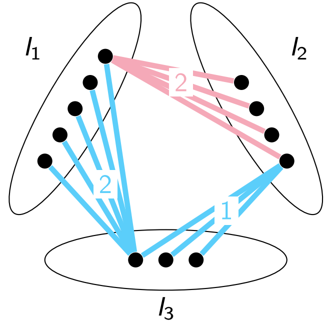

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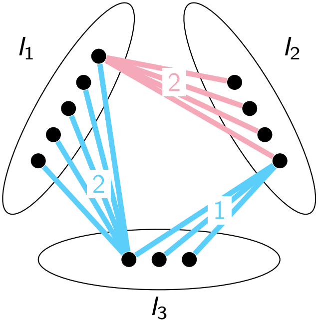

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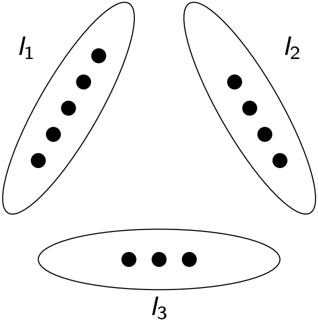

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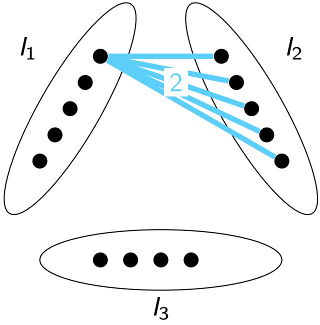

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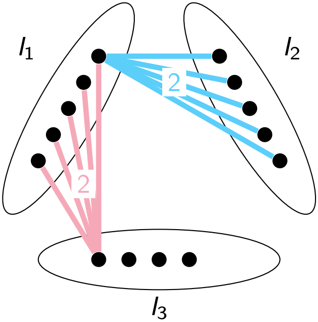

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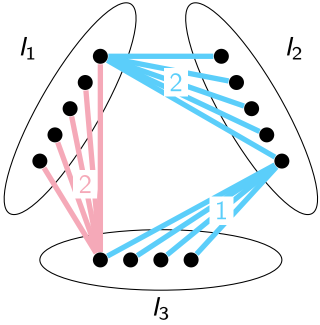

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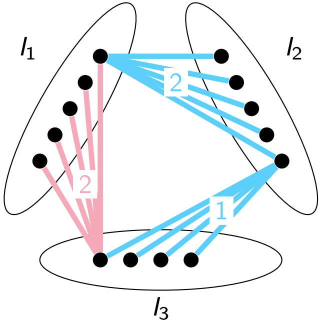

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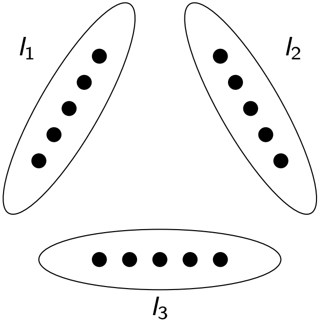

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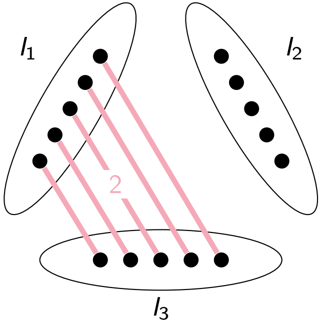

Proof of one of our Results

Theorem

If G is a complete k-partite graph other than K_2 and K_3, then G admits a strong $(2,2)$-colouring.

All vertices have a non-zero blue sum and no two adjacent vertices have red sum equal to 0.

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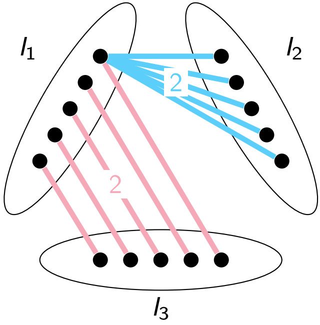

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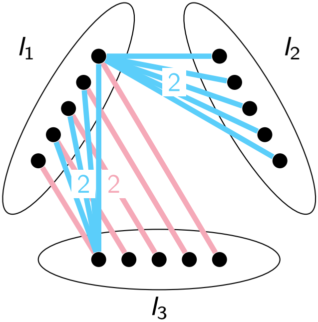

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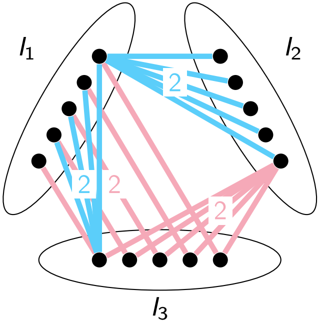

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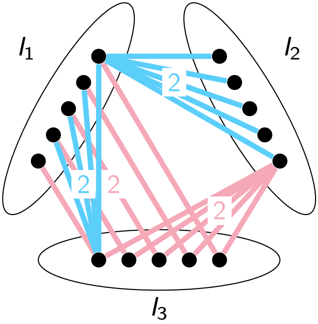

$I_1 : (2, 4n_1 - 2)$
$I_2 : (2n_1, 2n_1)$
$I_3 : (2n_1 + 2, 2n_1 - 2)$

Proof of one of our Results (part 2)

Theorem

If G is a complete 3-partite graph other than K_3, then G admits a strong $(2,2)$-colouring such that all vertices have a non-zero blue sum and no two adjacent vertices have red sum equal to 0.

Proof of one of our Results (part 2)

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Induction Hypothesis

If G is a complete k-partite graph for $k \geq 4$ odd (resp. even), then G admits a strong $(2,2)$-colouring such that all vertices have a non-zero blue (resp. red) sum and no two adjacent vertices have red (resp. blue) sum equal to 0.

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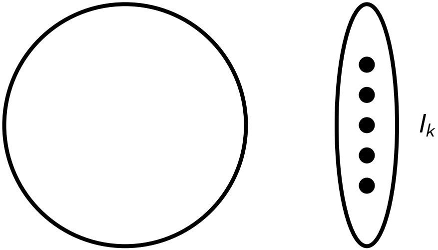

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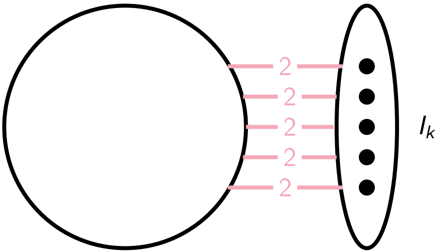

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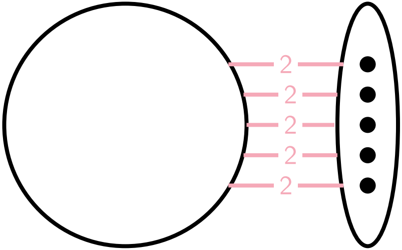

$I_k : (2(n_1 + \cdots + n_{k-1}), 0)$

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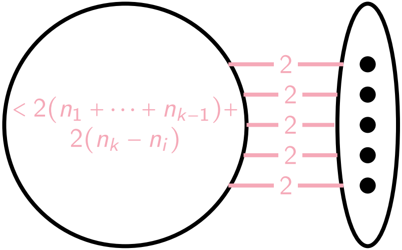

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