From Antimagic to Equitable Labellings

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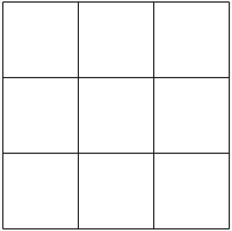
Magic labellings



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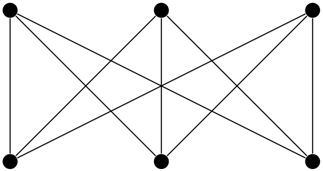

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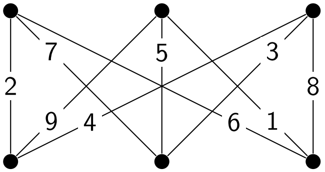

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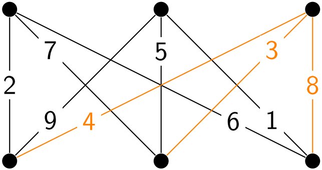

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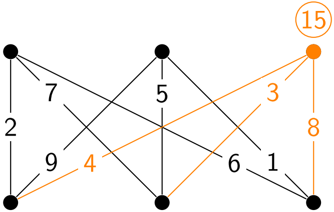

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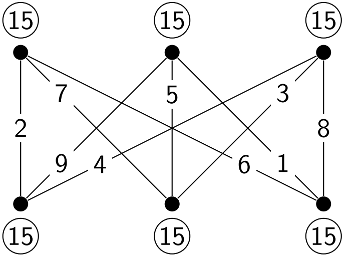

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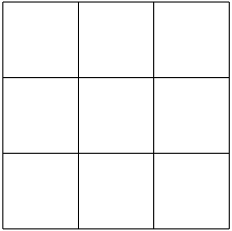

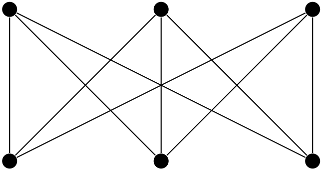

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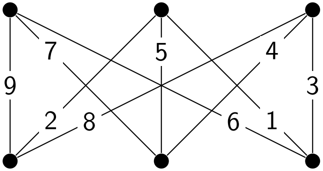

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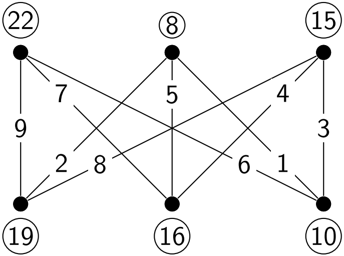

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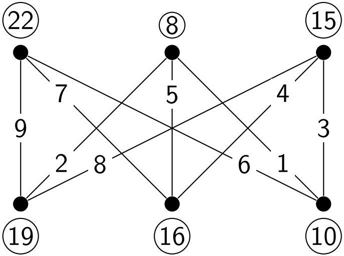

Definition

A *labelling* is said to be (locally) *distinguishing* if any pair of (adjacent) vertices have different *resulting sum*.

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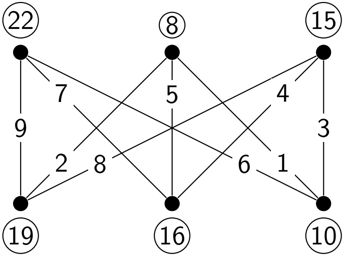

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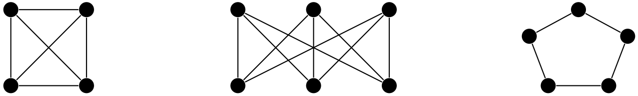
We will get back to all this terminology later.

Regular Graphs

Definition

A graph is *regular* if all its vertices have the same number of incident edges, called *degree*.

There are a lot of different regular graphs, for example:



Question

What is an *irregular* graph?

Irregular graphs

Observation

There are no simple graphs where every two vertices have different degree.

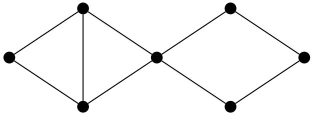
Local definition

A graph is *locally irregular* if every two adjacent vertices have different degree.

Irregularity Strength

Edges with multiplicity

We now allow two vertices to be joined by multiple edges.



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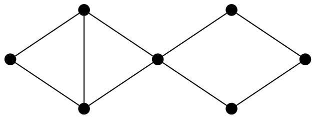

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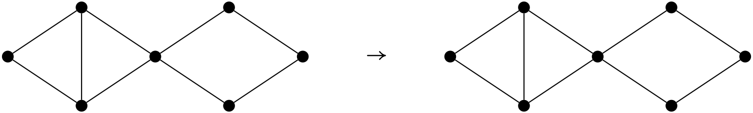

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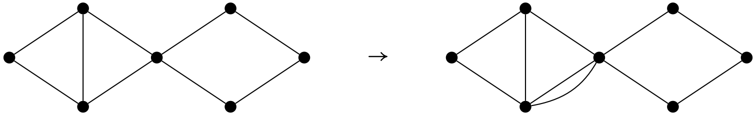

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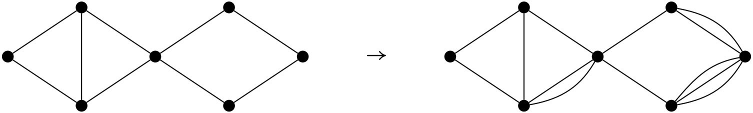

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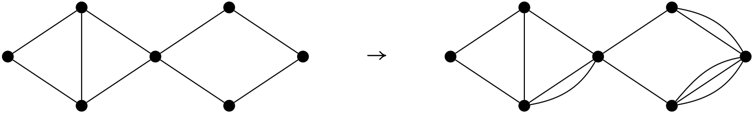

Definition

We call *irregularity strength* of a graph G the smallest multiplicity needed to make G irregular (every two vertices have distinct degree).

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Definition (Irregularity Strength)

We say a labelling ℓ is *distinguishing* if for every two vertices u and v of G, $\sigma_\ell(u) \neq \sigma_\ell(v)$.

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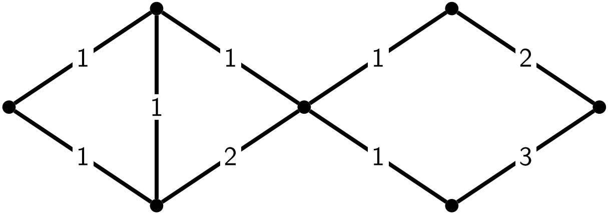

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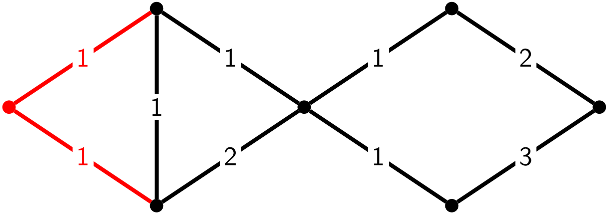

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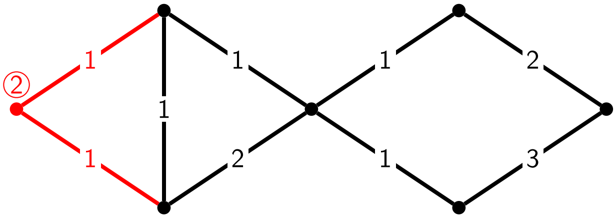

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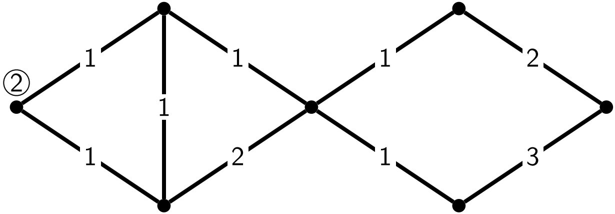

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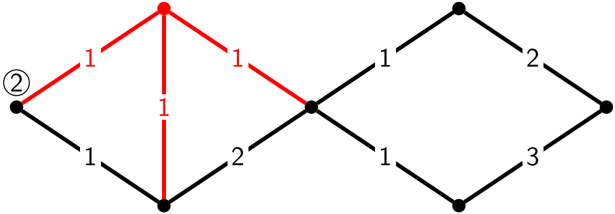

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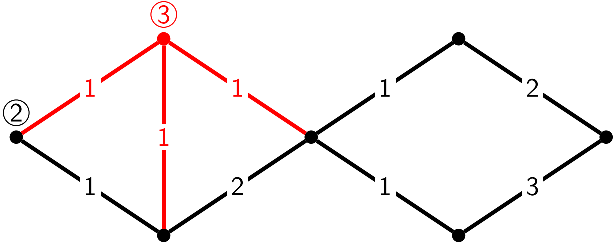

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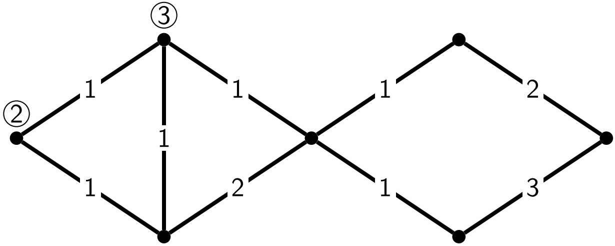

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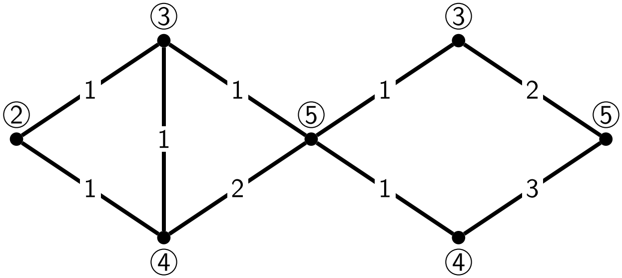

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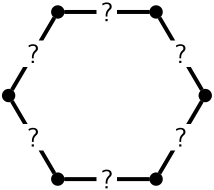

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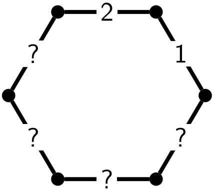

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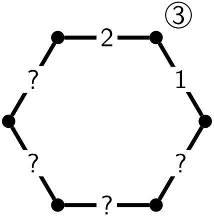

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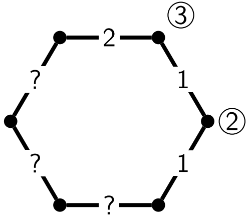

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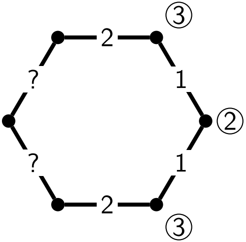

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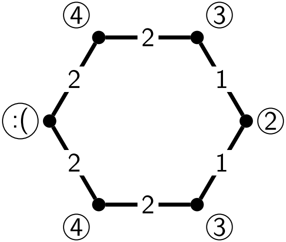

Conjectures and knowledge

These are the main conjectures and theorems in the field:

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1-2-3-4-5 Theorem (Kalkowski *et al.*, 2010)

All graphs admit a (distinguishing) 5-labelling.

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1-2 Problem (Ahadi *et al.*, 2013)

Given a graph G, deciding if G admits a 2-labellings is NP-complete.

Equitable Labellings

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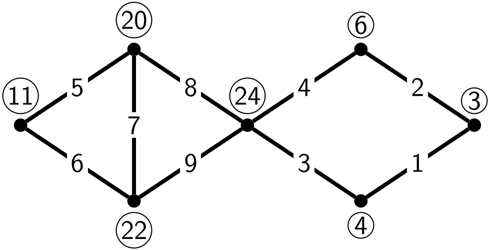
If G is a nice graph, then $\overline{\chi_\Sigma}(G) \leq |E(G)|$.

Examples

An equitable labelling which happens to be antimagic:



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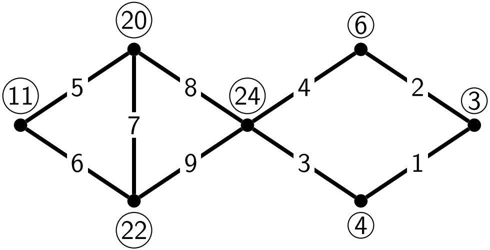
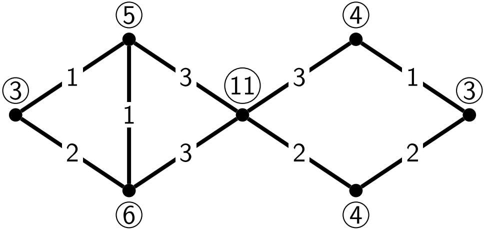


An equitable 3-labelling:



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Theorem (Bensmail, M., 2024+)

If G is a nice graph, then $\overline{\chi_\Sigma}(G) \leq \left\lfloor \frac{|E(G)|}{2} \right\rfloor + 2$.

Preliminary work

We consider the *sequence of labels* $L = (1, 1, 2, 2, \ldots, k+1, k+1, k+2, k+2)$ (where $k = \left\lfloor \frac{|E(G)|}{2} \right\rfloor$).

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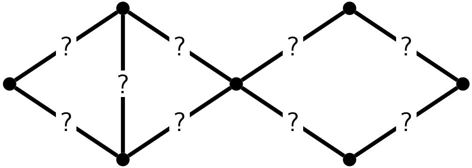

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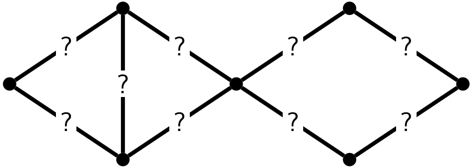

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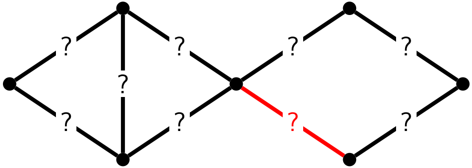

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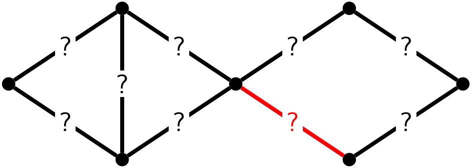

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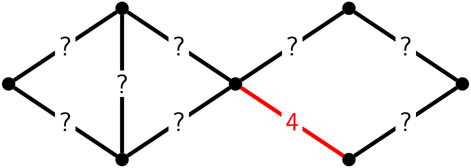

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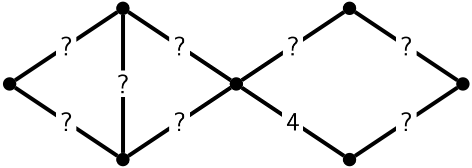

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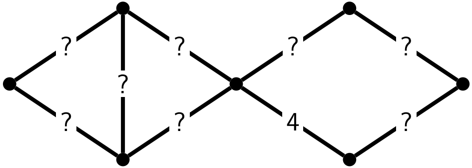

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Question

How do we carry the fact that some edges received a weight?

Weight function

Solution

We will keep the information by updating a *weight function*.

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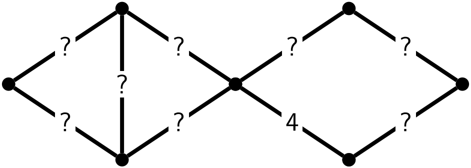

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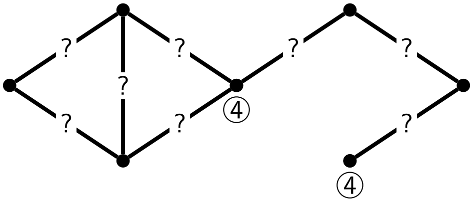

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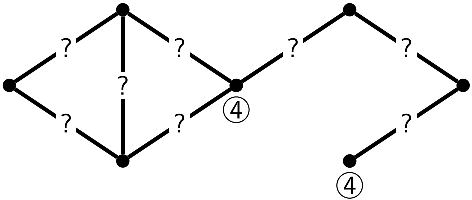

We now consider the weighted graph (G, c) where c is the weight of a vertex of G.

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- **Proceed by induction on the number of vertices.**
  We consider a vertex u and assign the labels of L to the edges incident to u to progressively to build a labelling ℓ, then remove u and the used labels.

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- **Build a partial labelling of the graph, and extend it.**
  We carry the contribution of the labelled edges through a weight function on the vertices adjacent to u.

Idea of the proof

- **Proceed by induction on the number of vertices.**
  We consider a vertex u and assign the labels of L to the edges incident to u to progressively to build a labelling ℓ, then remove u and the used labels.
- **Build a partial labelling of the graph, and extend it.**
  We carry the contribution of the labelled edges through a weight function on the vertices adjacent to u.

Problem

Once all the edges incident to u have been labelled, there is no way to change $\sigma_\ell(u)$.

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Problem

Maybe $G - u$ has a component isomorphic to K_2.

Idea of the proof

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- **Ensure that a vertex will have a resulting sum smaller than the vertex treate later in the induction.**
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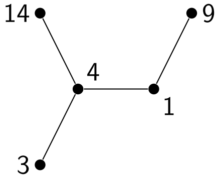
- **Handle exceptions on the way.**
  Identify and prevent problematic cases from arising while labelling.

Vertex of lowest potential

We want to find the vertex with the lowest potentential resulting sum:



For instance, consider $L = (6, 7, 7, 8, 8, 9, 9)$. Each vertex is annoted with its current weight.

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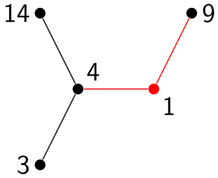

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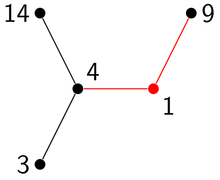

Here, the minimum possible resulting sum for this vertex is $1 + 6 + 7 = 14$.
For instance, consider $L = (6, 7, 7, 8, 8, 9, 9)$. Each vertex is annoted with its current weight.

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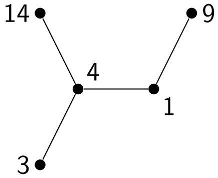

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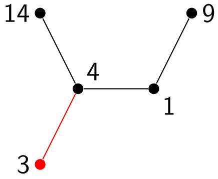

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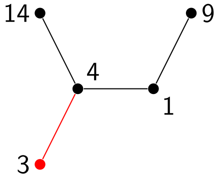

Here, the minimum possible resulting sum for this vertex is $3 + 6 = 9$.
For instance, consider $L = (6, 7, 7, 8, 8, 9, 9)$. Each vertex is annoted with its current weight.

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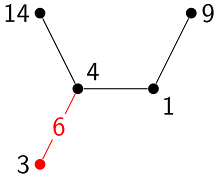

For instance, consider $L = (7, 7, 8, 8, 9, 9)$. Each vertex is annoted with its current weight.

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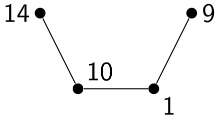
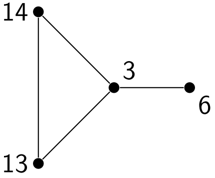


For instance, consider $L = (7, 7, 8, 8, 9, 9)$. Each vertex is annoted with its current weight.

Handling exceptions

Consider this weighted graph with $L = (1, 1, 2, 2, 3, 3, 4, 4)$:



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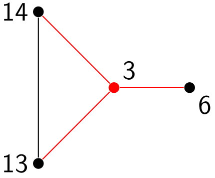
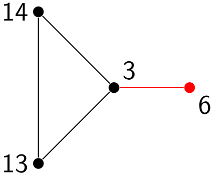


Handling exceptions

Consider this weighted graph with $L = ({\color{red}1}, 1, 2, 2, 3, 3, 4, 4)$:



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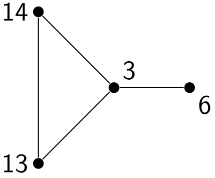

We choose a vertex of highest degree amongst vertices of lowest potential.

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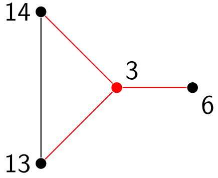

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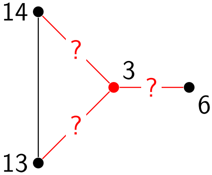

How to assign each label?

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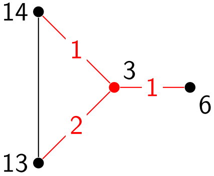

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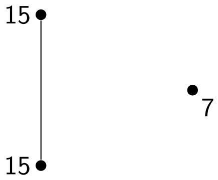
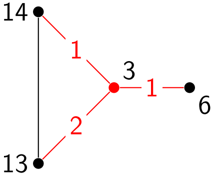


Problem

A component isomorphic to K_2 with constant weight cannot be labelled.

Handling exceptions

Consider this weighted graph with $L = (1, 1, 2, 2, 3, 3, 4, 4)$:



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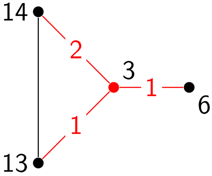

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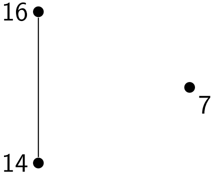

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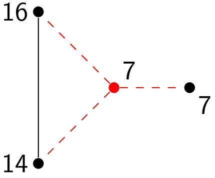

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Two adjacent vertices of lowest potential can be in conflict.

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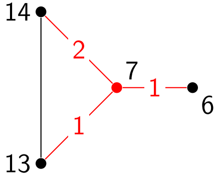

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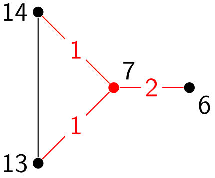

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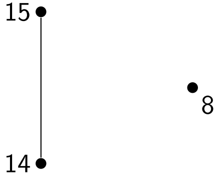

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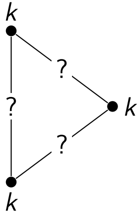

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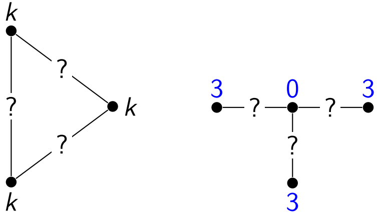

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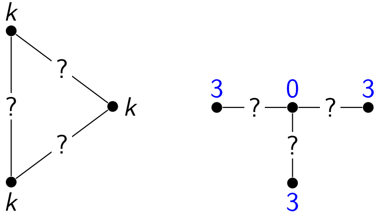

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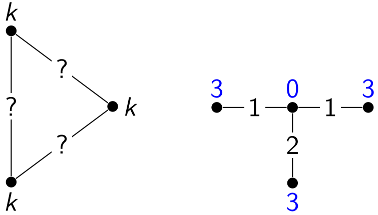

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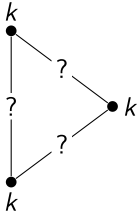

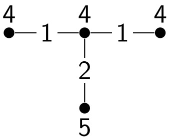

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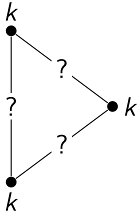

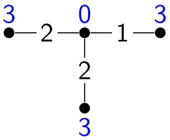

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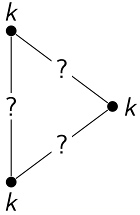

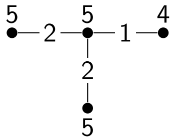

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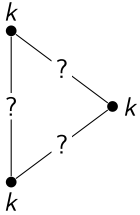

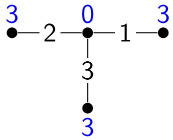

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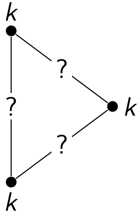

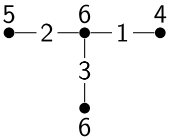

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Thank you for your attention!