

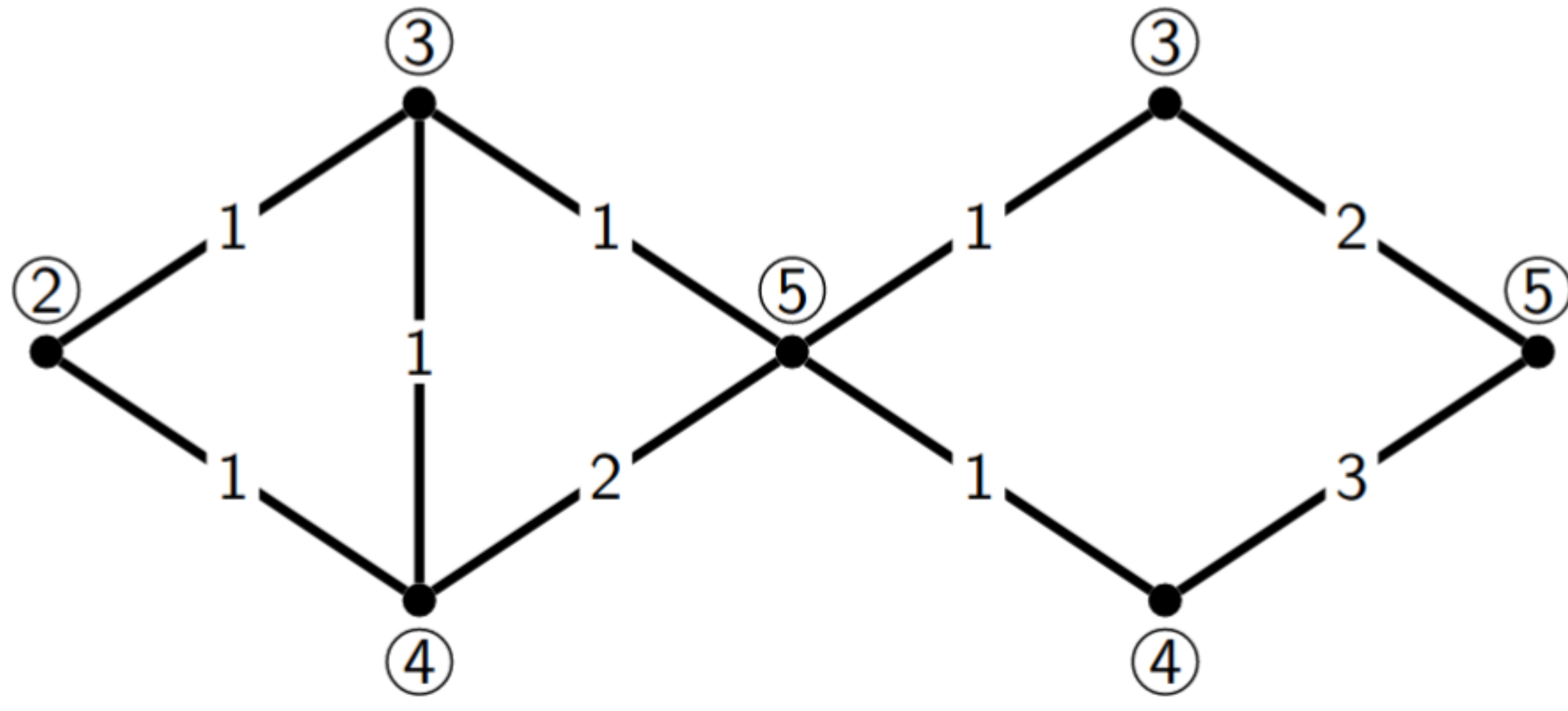
Adding direction constraints to the 1-2-3 Conjecture

Pierre-Marie Marcille - Julien Bensmail, Hervé Hocquard
CombAlgo - Univ. Bordeaux, LaBRI, France

LaBRI

k -labellings, resulting sums and χ_Σ

A **k -labelling** of a graph G is a function $\ell : E(G) \rightarrow \{1, \dots, k\}$.



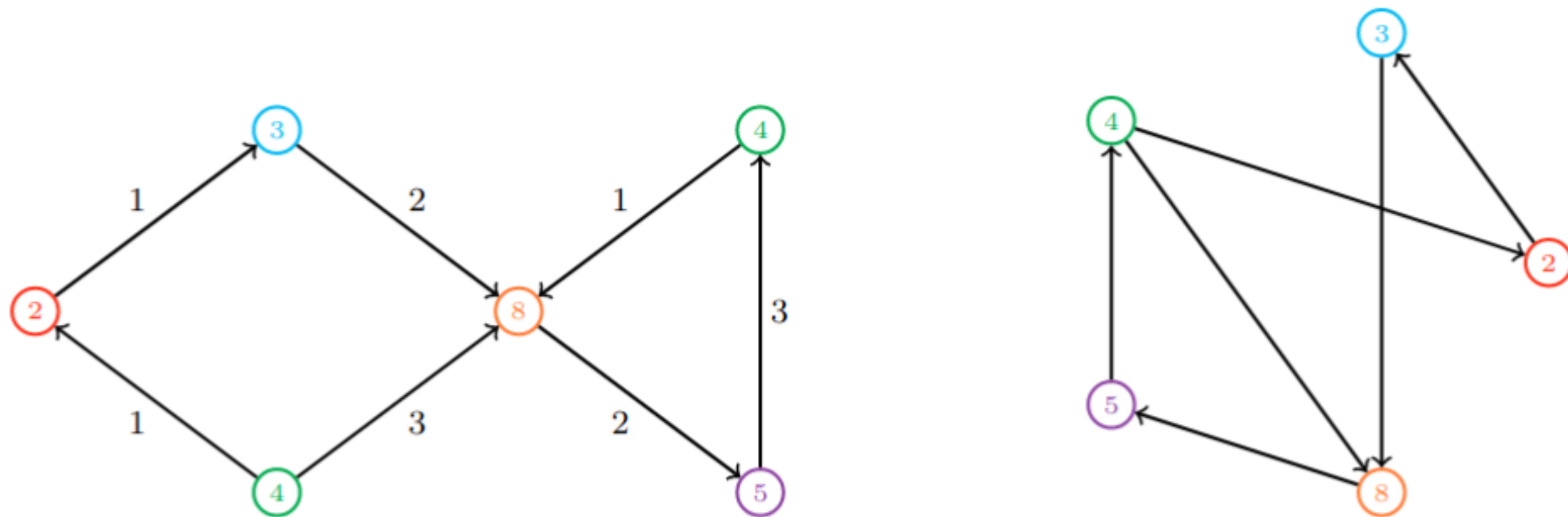
For every vertex, we can compute the **resulting sum** as the sum of the labels of its incident edges. A k -labelling is **distinguishing** if the resulting sums form a proper colouring. For a graph G , we denote by $\chi_\Sigma(G)$ the smallest k such that G admits a distinguishing k -labelling.

1-2-3 Conjecture [Karónski, Łuczak, Thomason, 2004]

If G is a graph with no isolated K_2 , then $\chi_\Sigma(G) \leq 3$.

Oriented colourings and χ_o

For an oriented graph \vec{G} , a k -colouring is **oriented** if it forms an homomorphism from \vec{G} to an oriented graph of order k .



We denote by $\chi_o(\vec{G})$ the smallest k such that \vec{G} admits an oriented k -colouring.

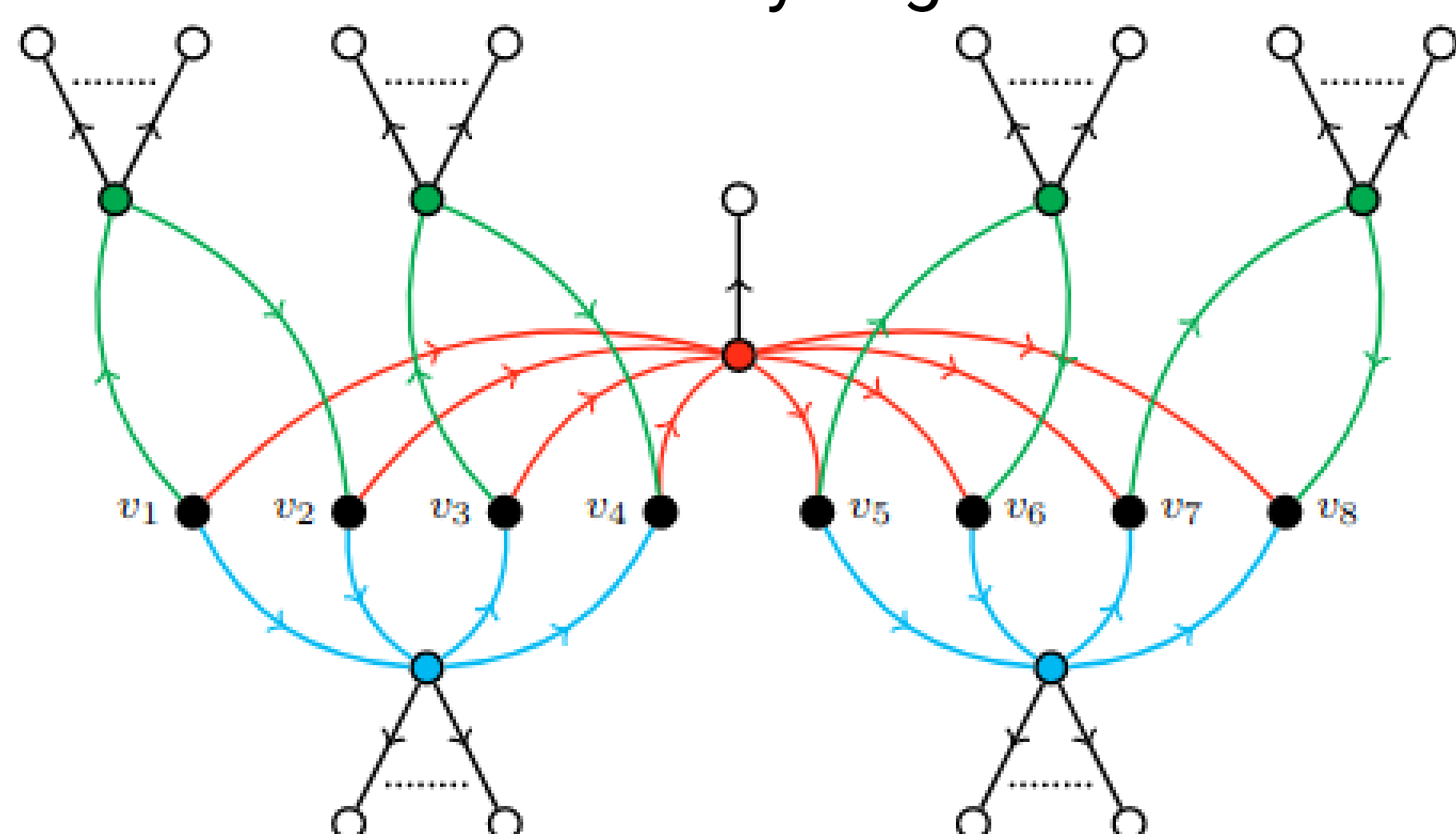
Oriented chromatic number and χ_Σ

We denote by $\chi_\Sigma(\vec{G})$ the smallest k such that there exists a k -labelling inducing an oriented colouring of \vec{G} .

Theorem: If G is a graph, then there exists an orientation \vec{G} of G such that $\chi_\Sigma(G) = \chi_\Sigma(\vec{G})$.

Connection with underlying graph

Theorem: There exist oriented graphs \vec{G} with $\chi_\Sigma(G) = 1$ and $\chi_\Sigma(\vec{G})$ arbitrarily large.



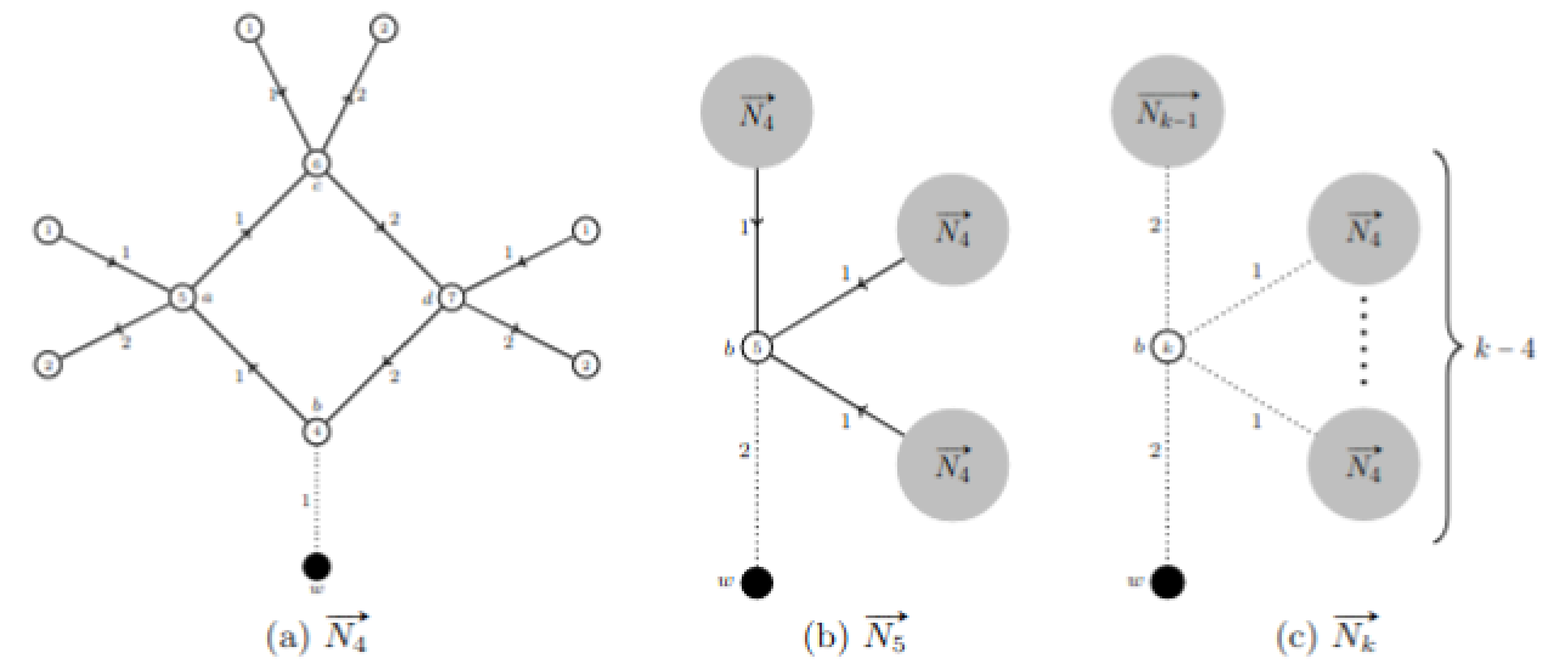
Global bound on χ_Σ

Theorem: There is no $k \geq 1$ such that $\chi_\Sigma(\vec{G}) \leq k$ for every \vec{G} .

Complexity result

Theorem: Deciding if $\chi_\Sigma(\vec{G}) \leq 2$ for a given \vec{G} is NP-hard.

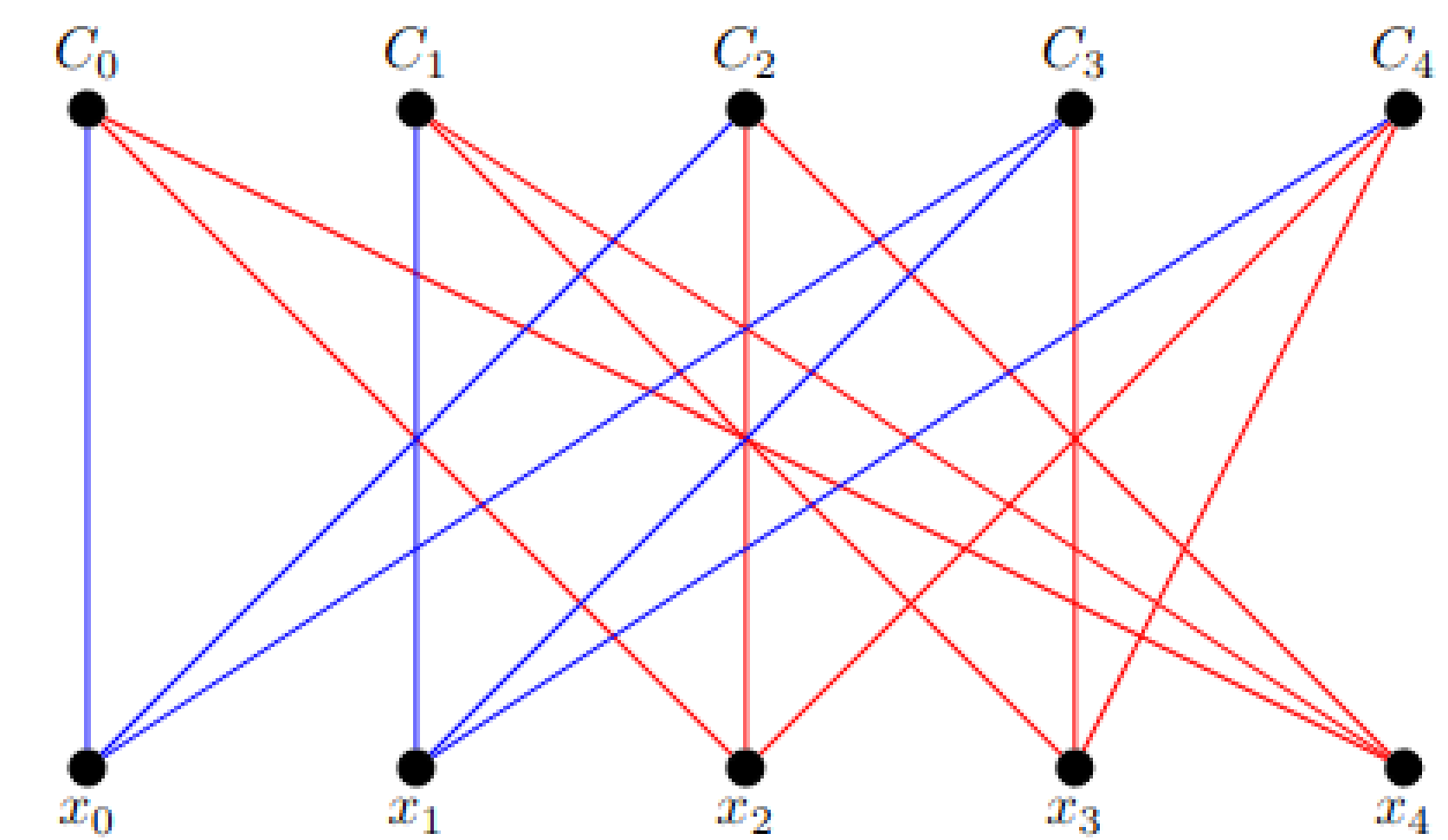
Gadgets



Lemma: The labels on the gadgets must be as shown.

Cubic Monotone 1-in-3 SAT

We consider a formula F instance of Cubic Monotone 1-in-3 SAT.



- ▶ To every clause, we attach eleven copies of \vec{N}_4 , one copy of \vec{N}_{20} , one copy of \vec{N}_{22} , and one copy of \vec{N}_{23} .
- ▶ To every variable, we attach three copies of \vec{N}_4 , one copy of \vec{N}_{11} , and one copy of \vec{N}_{12} .

This implies:

- ▶ The sum of the clause vertices must be 21; thus, two incident arcs receive label 1 and one receives label 2.
- ▶ The sum of the variable vertices must be 10 or 13; thus, either all incident arcs receive label 1, or they all receive label 2.

Consequences

- ▶ The instance resulting from the reduction is bipartite.
- ▶ Cubic Monotone 1-in-3 SAT is still NP-hard for planar formulae. Hence, the problem is still NP-hard when restricted to planar bipartite oriented graphs.

Conclusion and questions

- ▶ The value χ_Σ can be arbitrarily large.
- ▶ **Question:** How to express χ_Σ in general?
- ▶ We proved that deciding $\chi_\Sigma \leq 2$ is NP-hard.
- ▶ **Question:** What is the complexity of deciding $\chi_\Sigma \leq k$ for $k \geq 3$?