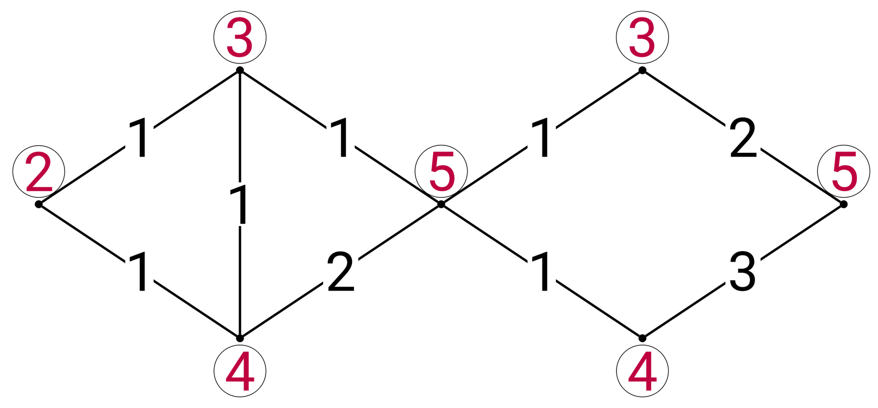
From Antimagic to Equitable Labellings

Clara Marcille - Julien Bensmail, Hervé Hocquard
CombAlgo - Univ. Bordeaux, LaBRI, France

LaBRI

Edge labellings

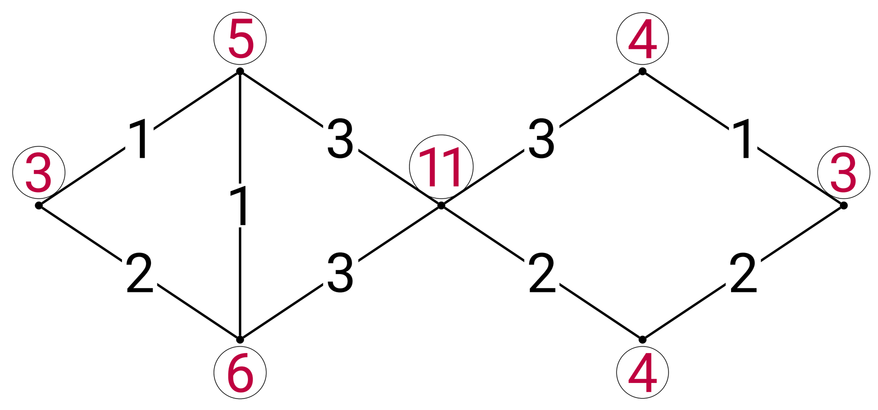
A *k-labelling* of a graph G is a function $\ell : E(G) \to \{1, \ldots, k\}$.
We call *resulting sum* (relative to a labelling ℓ) of a vertex u the sum of labels on edges incident to u. We denote it $\sigma_\ell(u)$.



We say a labelling ℓ is *proper* if for every two adjacent vertices u and v of G, $\sigma_\ell(u) \neq \sigma_\ell(v)$.

Equitable labellings

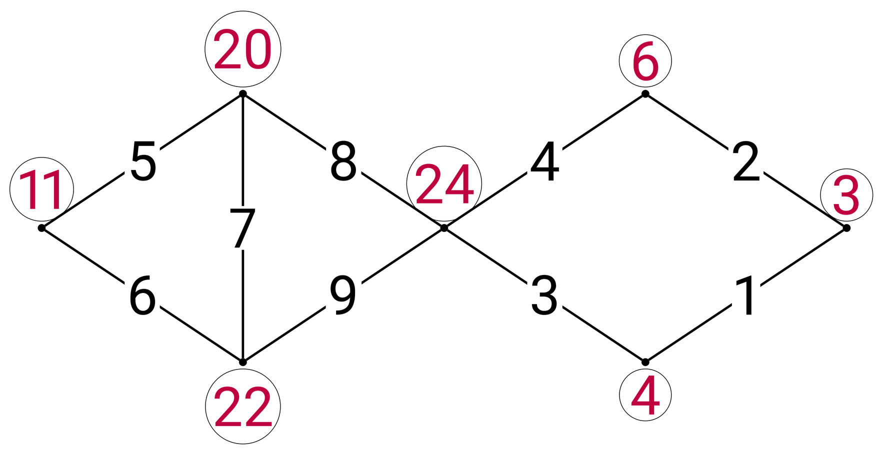
We say a labelling is *equitable* if every pair of labels appear the same number of time.



A labelling can be both equitable and proper!

Antimagic labellings

We say a labelling ℓ is *locally antimagic* if ℓ is both proper and a bijection between $E(G)$ and $\{1, \ldots, |E(G)|\}$.



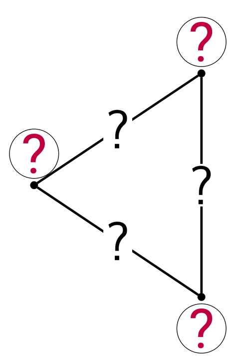
A locally antimagic labelling is a special case of proper equitable $|E(G)|$-labelling!

Local Antimagic Theorem

Local Antimagic Theorem, Lyngie and Zhong, 2018:
If G is a graph, then G admits a locally antimagic labelling.
This theorem, with the definitions introduced before, implies the following:
Corollary:
If G is a graph, then G admits a proper equitable $|E(G)|$-labelling.
Note: we only consider graphs with no component isomorphic to K_2.

Problem statement

We denote by $\overline{\chi_\Sigma}(G)$ the smallest $k \geq 1$ such that G admits equitable proper k-labellings. Is there a better bound than $\overline{\chi_\Sigma}(G) \leq |E(G)|$?
Conjecture, Bensmail, Fioravantes, Mc Inerney, Nisse, 2021:
If G is a graph different from K_4, then $\overline{\chi_\Sigma}(G) \leq 3$.



You can try to label this C_3 in an equitable way!
Is it possible to have a 2-equitable labelling?
Does that mean we can not hope for better than an Antimagic labelling?

Our contribution

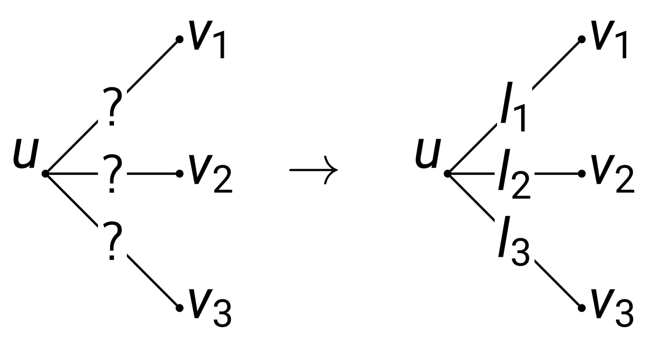
If G is a graph, then $\overline{\chi_\Sigma}(G) \leq \left\lfloor \frac{|E(G)|}{2} \right\rfloor + 2$.

Idea of the proof

- **Proceed by induction on the number of vertices.**
  We try to assign the labels of the sequence $(1, 1, 2, \ldots, k+2, k+2)$ where $k = \left\lfloor \frac{|E(G)|}{2} \right\rfloor$ to progressively to build ℓ.
- **Build a partial labelling of the graph, and extend it.**
  We carry the contribution of the labelled edges through a weight function on the vertices.
- **Ensure that a vertex will have a resulting sum distinct of its neighbours.**
  We find the vertex of lowest *potential* resulting sum.
- **Handle exceptions on the way.**
  Identify and prevent problematic cases from arising while labelling.

The weight function c

We start with weight $c = 0$ everywhere. When we treat a vertex u, and assign labels $l_1, \ldots, l_{d(u)}$ to the edges incident to u, and we update c.

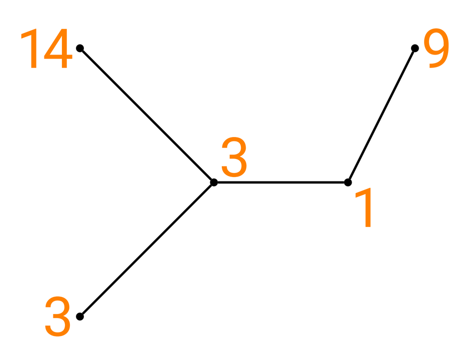


$c(v_1) += l_1$
$c(v_2) += l_2$
$c(v_3) += l_3$

We then remove u and proceed by induction. But how exactly do we choose the l_i's?

Treating a vertex

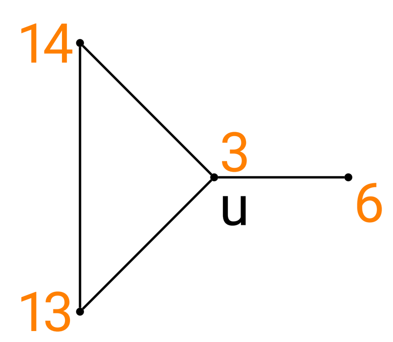
We choose u to have the smallest value $c(u) + x$, where x is the sum of the $d(u)$ smallest labels not assigned yet, so that u will be distinguished.



Here, assume the sequence of labels is $l = (6, 7, 7, 8, 8, 9, 9)$.
We annoted each vertex with its weight at this step of induction.
Find the next vertex to treat!

Problematic cases

To be able to call the induction, we must make sure that removing u will not yield a problematic case.



Here, assume the sequence of labels is $l = (1, 1, 2, 2, 3, 3, 4, 4)$.
Try to assign the labels $(1, 1, 2)$ to the edges incident to u!
Only one way is correct!

Conclusion

- This kind of problem is one of the numerous problems on graph labellings.
- We essentially improved the best known general upper bound on $\overline{\chi_\Sigma}$ by about a factor 2.
- We are still far from a constant bound as it was conjectured, the main problem remains open to this day.