

RECONFIGURING DOMINATING SETS

UNDER TOKEN SLIDING

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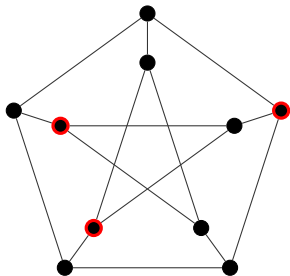
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University of Bordeaux

Domination

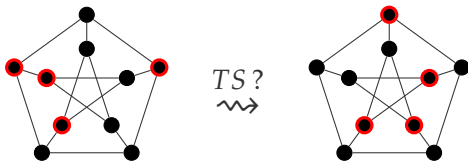
Definition

A dominating set in a graph $G = (V, E)$ is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one member of D .

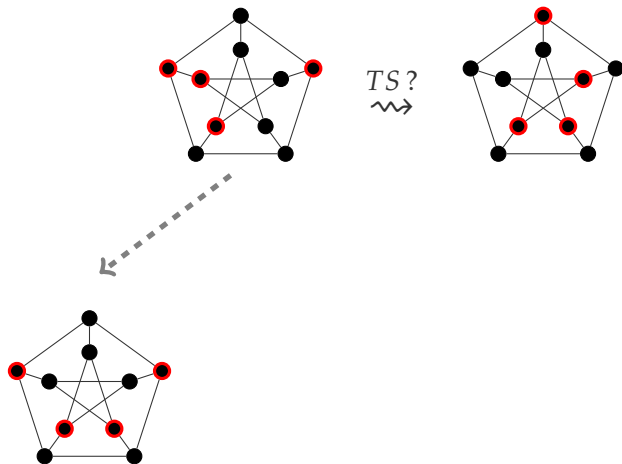


Graph reconfiguration

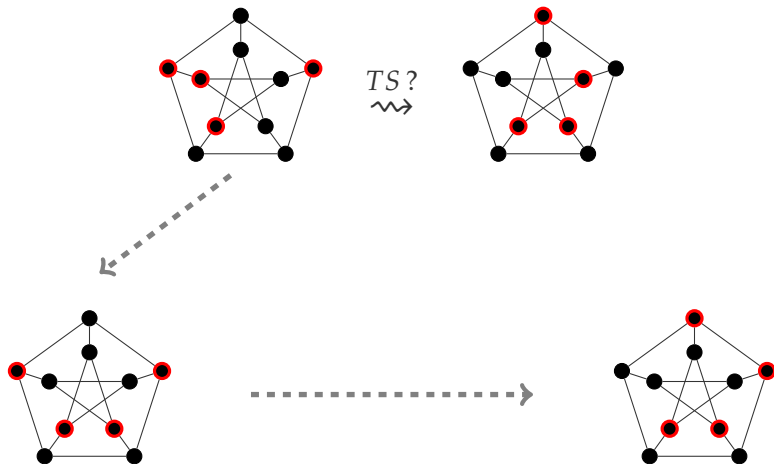
Rule : each intermediate solution must be a dominating set, we are only allowed to slide a token along an edge !



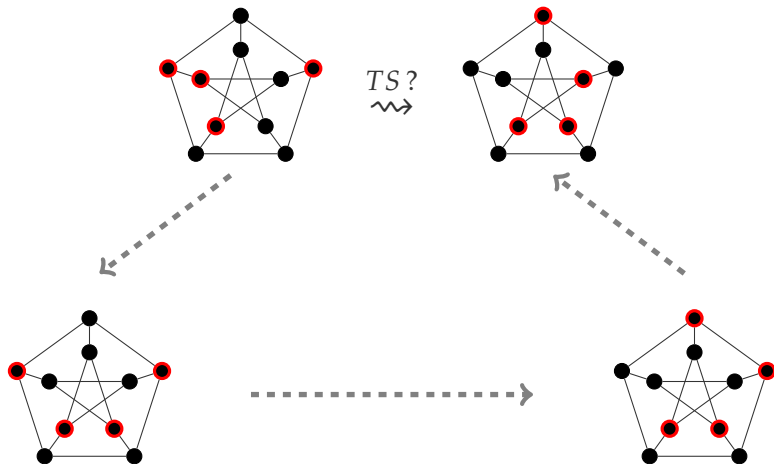
Graph reconfiguration



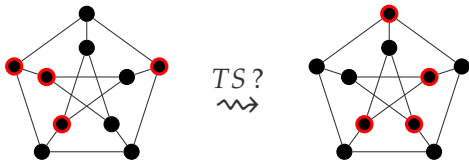
Graph reconfiguration



Graph reconfiguration

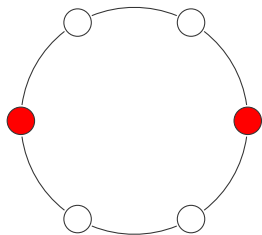


Graph reconfiguration

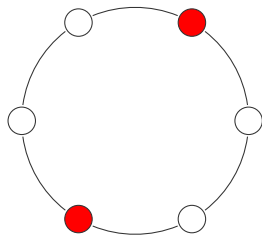


Example

⚠ not always possible !



TS?



Dominating Set Reconfiguration

Input : a graph G , two dominating sets A and B of G

We want : to transform step-by-step A into B ; each intermediate solution must be a dominating set.

Elementary operations

- *token addition and removal* (TAR)
- *token jumping* (TJ)
- *token sliding* (TS)

Notations

Let G be a graph and A, B be two dominating sets of size k of G .

- $A \overset{TAR}{\rightsquigarrow} B$: we can reconfigure A into B with the TAR model and each solution is of size at most $k + 1$.
- $A \overset{TJ}{\rightsquigarrow} B$: we can reconfigure A into B with the TJ model.

Lemma

Let G be a graph and A and B be two dominating sets of G of size k . We have $A \overset{TAR}{\rightsquigarrow} B$ iff $A \overset{TJ}{\rightsquigarrow} B$.

Proof : adapted from ¹ (Theorem 1)

- $TJ \Rightarrow TAR$: easy. Replace each move $u \rightsquigarrow v$ by :
 - first add v
 - remove u
 - \Rightarrow each solution has size at most $k + 1$
- We double the length of the sequence

¹M. Kaminski, P. Medvedev, and M. Milanic. Complexity of independent set reconfigurability problems. *Theoretical Computer Science*, 439 :9 – 15, 2012.

- $TAR \Rightarrow TJ$
 - TAR-sequence of length $2n$ with configurations of size k or $k + 1$
 - Alternation of addition of v followed by a removal of u
 - Replace by $u \overset{TJ}{\rightsquigarrow} v$
 - \Rightarrow get a TJ-sequence of length n
 - TAR-sequence of length $2n$ with configurations of any size
 - There exists a subsequence which consists in the removal of a vertex u followed by the addition of a vertex v
 - If $u = v \rightarrow$ remove the subsequence
 - Otherwise, switch the order
 - (Possibly reiterate the process)

Theorem [Haddadan *et al.*]

Reconfiguring dominating sets under $\text{TAR}(k + 1)$ is PSPACE-complete.

The problem is in PSPACE² (Theorem 1).

Lemma

Reconfiguring dominating sets under $\text{TS}(k)$ is PSPACE-complete in split graphs.

¹Ito, T., Demaine, E.D., Harvey, N.J.A., Papadimitriou, C.H., Sideri, M., Uehara, R., Uno, Y.: On the complexity of reconfiguration problems. *Theoretical Computer Science* 412, pp. 1054–1065 (2011)

Complexity proof

Proof : reduction from TJ in some graph $G \rightarrow$ TS in the corresponding split graph.

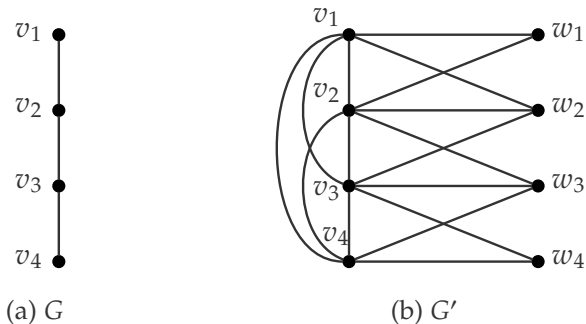


Figure: Graph G and the corresponding split graph G'

Definitions

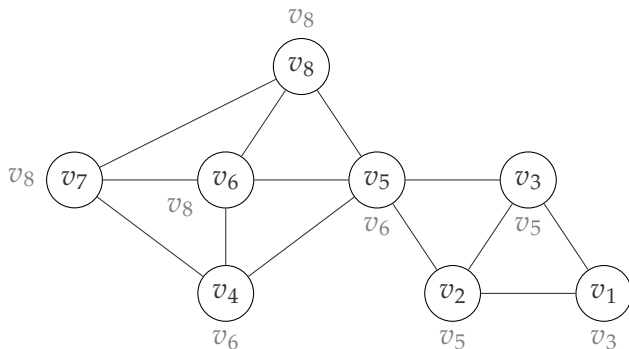
- $G = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$
- $G_i = G[\{v_i, v_{i+1}, \dots, v_n\}]$
- v *maximum neighbor* of u :
 - $v \in N[u]$
 - $\forall w \in N[u] : N[w] \subseteq N[v]$
 - $\Rightarrow v$ is adjacent to all the vertices at distance ≤ 2 from u
- *maximum neighborhood ordering* (mno) : ordering on the vertices such that v_i has a maximum neighbor in G_i

A *dually chordal* graph is a graph which has a maximum neighborhood ordering.

Lemma [Dorbec, Kosmrlj, Renault]

Let G be a dually chordal graph. There exists a maximum neighborhood ordering v_1, v_2, \dots, v_n of G such that if the only maximum neighbor of v_i in G_i is itself, then v_i is an isolated vertex in G_i . We call such a mno a *proper mno*.

Example of dually chordal graph



Minimum dominating sets reconfiguration

Let A and B be two *minimum* dominating sets of a dually chordal graph G . We want to reconfigure A into B .

Proof (sketch) :

- We distinguish a minimum dominating set
→ *canonical* dominating set denoted C
- We show that both A and B reconfigure into C
- Reversible operation \Rightarrow one can reconfigure A into B

Algorithm to compute the canonical dominating set

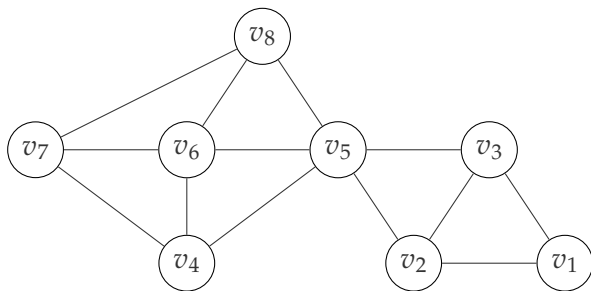
Input : a dually chordal graph G , a proper mno

Output : a canonical dominating set of G

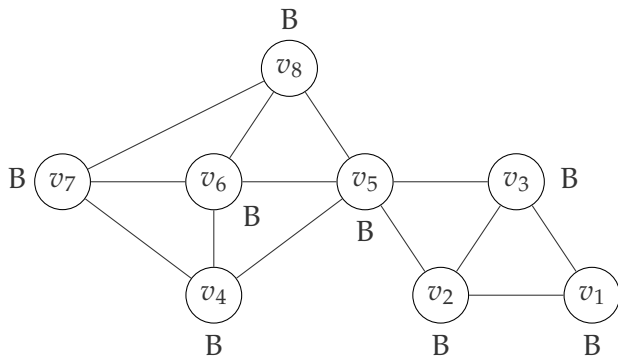
1. Label all vertices with BOUNDED
2. For i from 1 to $\gamma(G)$
 - If v_i is BOUNDED
 - Mark vertex v_i
 - Label $mn(v_i)$ with REQUIRED
 - For each vertex $v \in N(mn(v_i))$ which is **not** REQUIRED, label it FREE

\Rightarrow The vertices labeled REQUIRED form a dominating set.

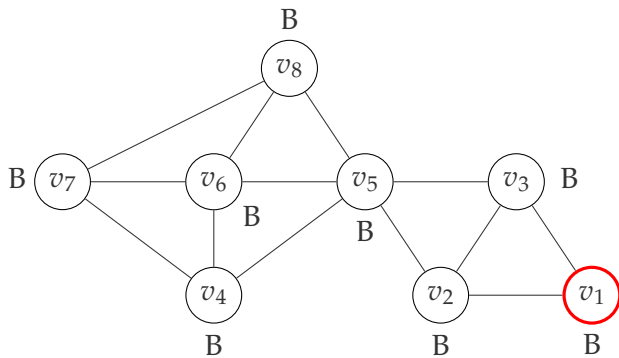
Example



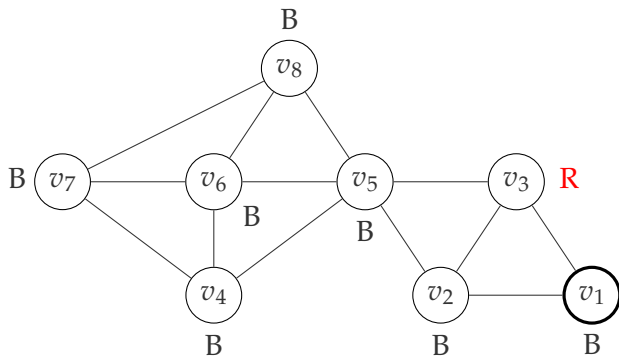
Example



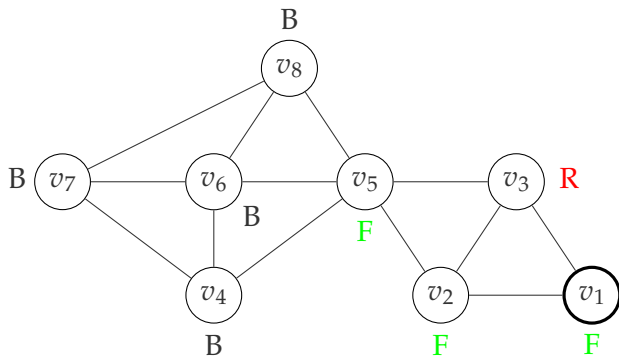
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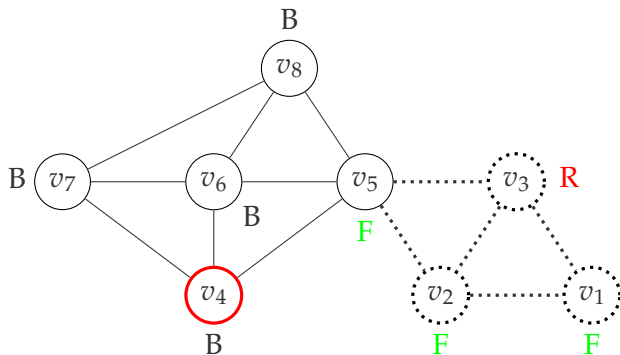
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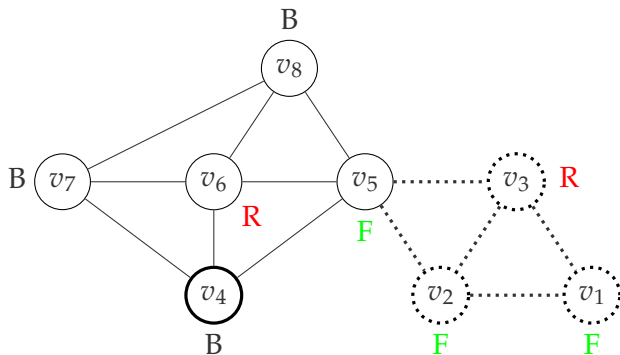
Example



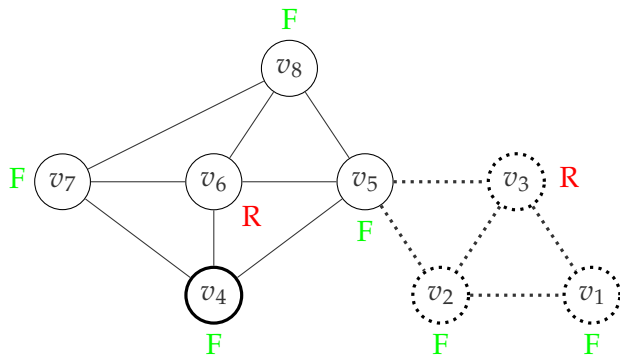
Example



Example



Example



Reconfiguration algorithm

- Let $C = \{c_1, c_2, \dots, c_{\gamma(G)}\}$ be the canonical dominating set
- Let $m_1, m_2, \dots, m_{\gamma(G)}$ be the marked vertices (by the previous algorithm)
- Let D be the minimum dominating set that we want reconfigure into C
- Let x_i be a vertex in $N[m_i] \cap D$

Proof (sketch) :

- **$i = 1$:**
 - $\forall w \in N[m_1], N[w] \subseteq N[c_1]$
 - In particular : $N[x_1] \subseteq N[c_1]$
 - $\Rightarrow (D \setminus \{x_1\}) \cup \{c_1\}$ is a dominating set of G
 - $(x_1, c_1) \in E(G)$

Reconfiguration algorithm

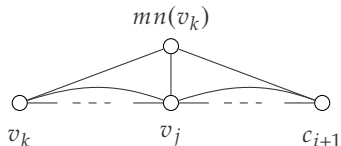
- $i \Rightarrow i + 1$
- Suppose it is true at rank i i.e.
 $(D \setminus \{x_1, \dots, x_i\}) \cup \{c_1, \dots, c_i\}$ is a dominating set of G
- We denote $v_j = m_{i+1}$ and $v_k = x_{i+1}$
- We distinguish two cases :
 - $v_k \geq v_j$:
 - all the vertices before v_j (in the mno) are already dominated
 - $N_j[x_{i+1}] \subseteq N_j[c_{i+1}]$
 - $\Rightarrow (D \setminus \{x_1, \dots, x_i, x_{i+1}\}) \cup \{c_1, \dots, c_i, c_{i+1}\}$ is also a dominating set of G
 - $\Rightarrow (x_{i+1}, c_{i+1}) \in E(G)$

Reconfiguration algorithm

○ $i \Rightarrow i + 1$

○ $v_k < v_j$:

- $v_k \notin \{c_1, c_2, \dots, c_i\}$ (because v_j is labeled BOUNDED)
- all the vertices before v_j (still in the mno) are already dominated
- If $(v_k, c_{i+1}) \in E(G)$: nothing to do
- Otherwise, the situation is the following :



- We can slide v_k to c_{i+1} in two steps (via $mn(v_k)$).

Conclusion

- Problem PSPACE-complete in some graph classes...
 - Split graphs
 - Planar graphs
 - Bounded treewidth graphs
- ... but polynomial in other ones
 - Interval graphs (for any $k \geq \gamma(G)$)
 - Dually chordal (for $k = \gamma(G)$)
 - Cographs
- Find a class for which computing a minimum dominating set is hard but for which reconfiguration can be done in polynomial time.

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Thank you for your attention!

$TAR(k) \not\Rightarrow TAR(k + 1)$

