

# PRICE OF CONNECTIVITY

FOR DOMINATION

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Paul OUVRARD

July 10, 2018

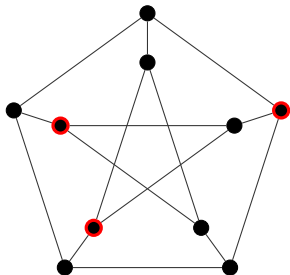
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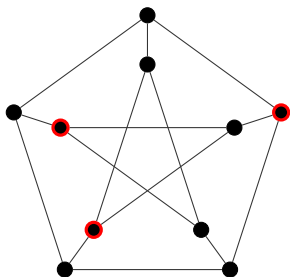
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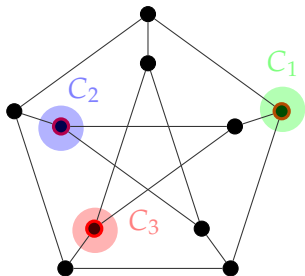
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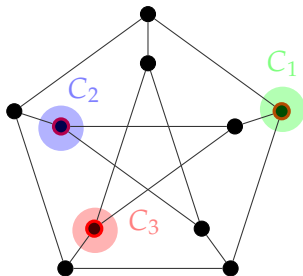
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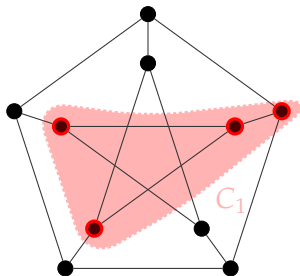
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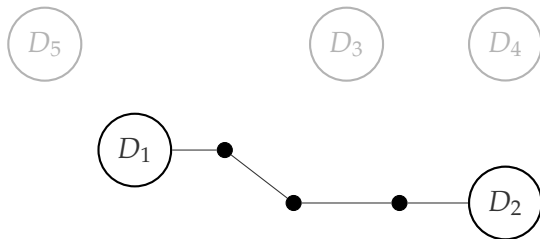
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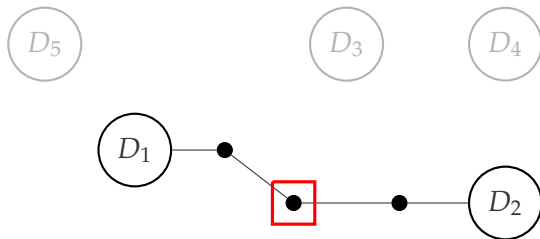
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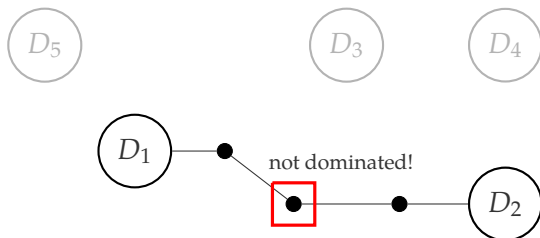
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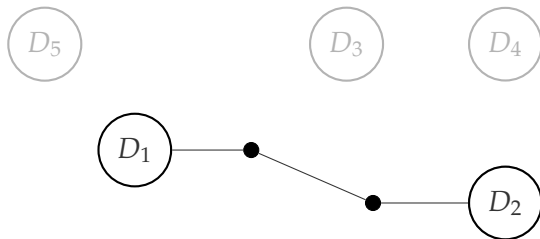
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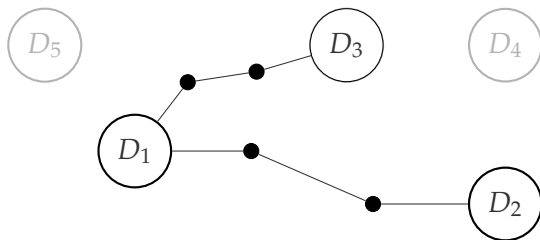
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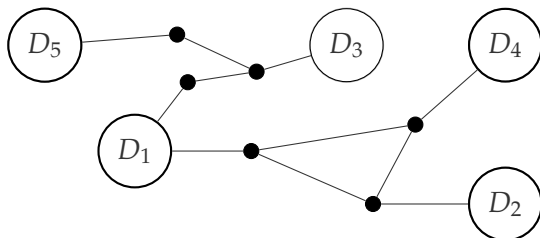
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$$\gamma_c(P_n) = n - 2$$



## PoC-Perfect graph

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## Theorem [Zverovich '03]

For every graph  $G$ ,  $G$  is PoC-Perfect iff  $G$  is  $(P_5, C_5)$ -free.

## PoC-Near-Perfect graph

A graph  $G$  is **PoC-Near-Perfect** with threshold  $t$  if for every **induced subgraph**  $H$  of  $G$ ,  $\gamma_c(H) \leq t \cdot \gamma(H)$ .

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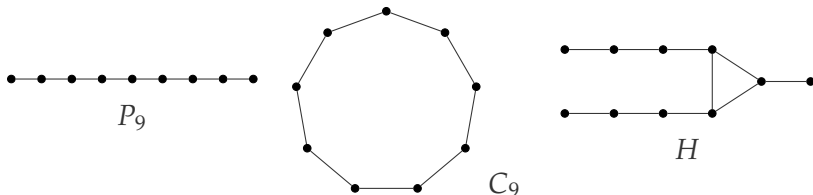
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Conjecture: [Camby and Schaudt '14]

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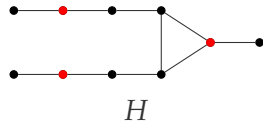
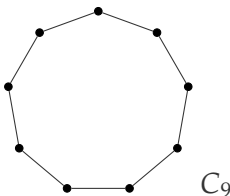
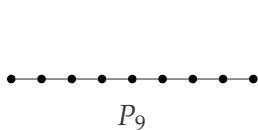
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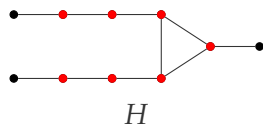
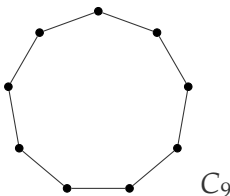
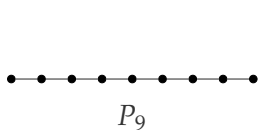
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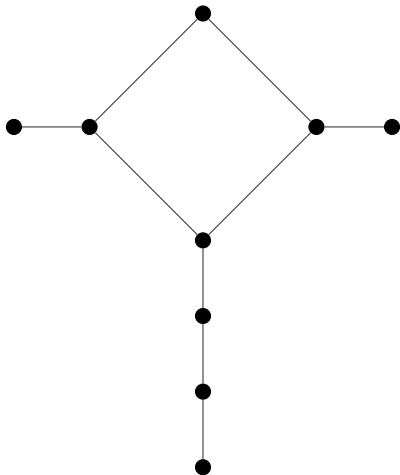
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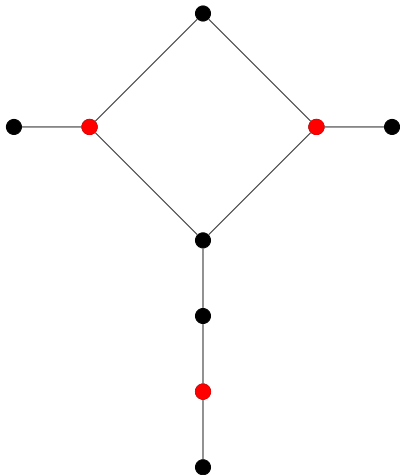
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- $\ell$  the largest integer s.t. there exists a dominating set  $D$  s.t.
  - $G[D]$  has  $g - \ell$  cc;
  - $|D| \leq \gamma(G) + \ell$ ;
  - with some notion of heredity.

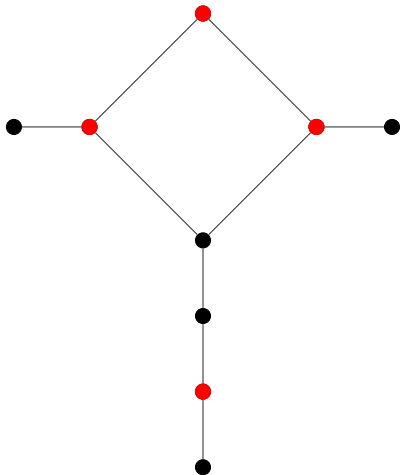
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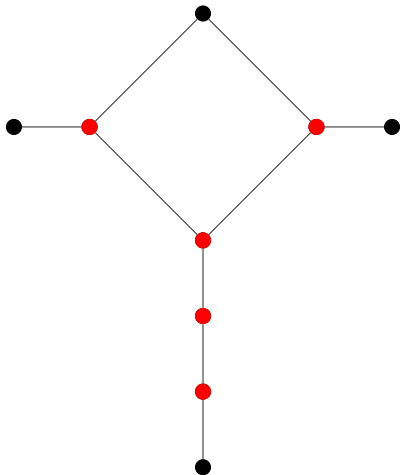


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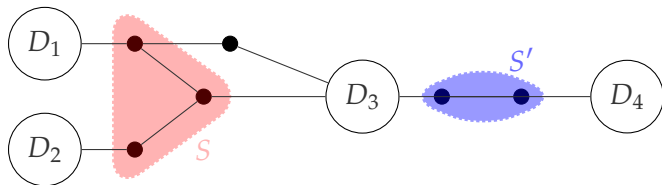
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## Definition

Semi-relevant set  $S \subseteq V \setminus D$  s.t.  $G[D \cup S]$  has  $g - \ell - |S| + 1$  cc.

$\Rightarrow$  every semi-relevant set is connected.

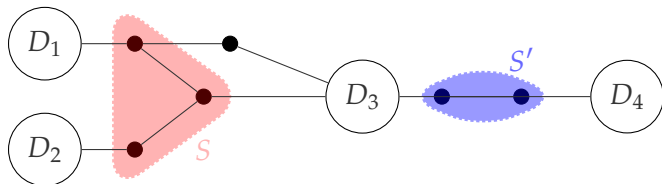
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- $A(S) = \{D_1, D_2, D_3\}$ ;
- there is a bijection between  $S$  and  $A(S)$ ;
- for every  $i$ , there is  $u_i \in N[D_i]$  with no neighbor in  $S$ ...

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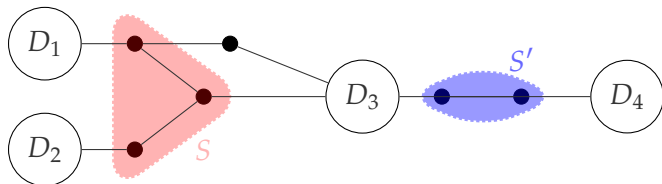
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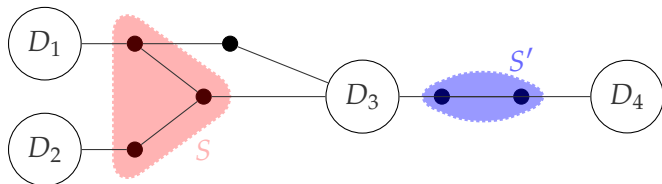
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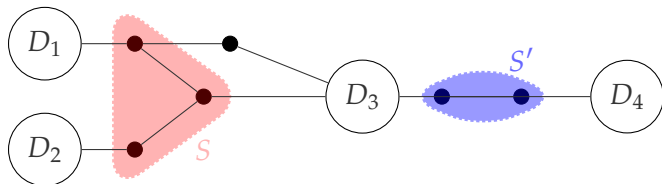
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- $D_t$ : a cc. with no neighbor in  $S \cup S'$ ;
- $\text{dist}(D_p, D_t) = 3$  for some  $D_p \in A(S) \cup A(S')$ ;

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$D' = D \cup S \cup S'$  is a connected dominating set of size:

$$\begin{aligned} |D'| &= |D| + |S| + |S'| \\ &= |D| + |A(S)| + |A(S')| \\ &= |D| + |A(S) + A(S')| + 1 \\ &= |D| + g - \ell + 1 \\ &\leq \gamma + \ell + g - \ell + 1 \end{aligned}$$

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$\Rightarrow \gamma_c(G) = 2\gamma(G) + 1$ .



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- If  $D^* = D' \setminus \{w_1\}$  is a connected dominating set:  
 $|D^*| = 2\gamma(G)$  ☺
- If not,  $w_1$  has a private neighbor in  $D^* \Rightarrow G$  contains a forbidden induced subgraph ☺.

**Thank you for your attention!**

# Some extra...

$n$	4	5	6	7	8	9	10	11	12	13	14	15
$\gamma(P_n)$	2	2	2	3	3	3	4	4	4	5	5	5
$\gamma_c(P_n)$	2	3	4	5	6	7	8	9	10	11	12	13
PoC	1	$\frac{3}{2}$	2	$< 2$	2	$\frac{7}{3}$	2	$< \frac{7}{3}$	2.5	2.2	2.4	2.6