

PRICE OF CONNECTIVITY

FOR DOMINATION

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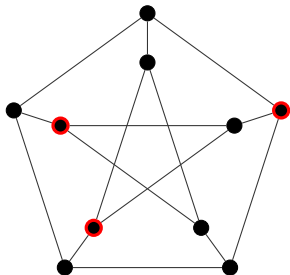
Graphs and Matroids Seminar
University of Waterloo



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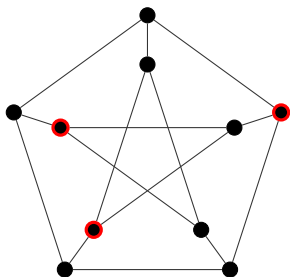
Dominating set D

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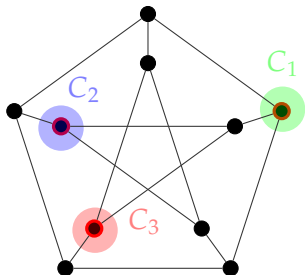
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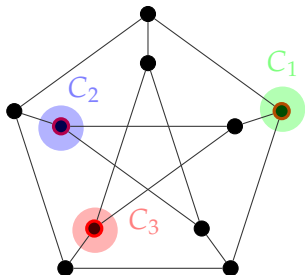
Connected dominating set D

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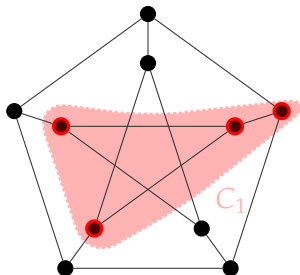
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Price of Connectivity for domination [Cardinal, Levy '08]

Let G be a graph. We define $\text{PoC}(G)$ as $\frac{\gamma_c(G)}{\gamma(G)}$.

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- $\gamma_c(G) \leq 3\gamma(G) - 2$
⇒ asymptotically sharp for paths and cycles...

PoC-Perfect graph

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Theorem [Zverovich '03]

For every graph G , G is PoC-Perfect iff G is (P_5, C_5) -free.

PoC-Near-Perfect graph

A graph G is **PoC-Near-Perfect** with threshold t if for every **induced subgraph** H of G , $\gamma_c(H) \leq t \cdot \gamma(H)$.

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Theorem: $t = 3/2$ [Camby and Schaudt '14]

A graph G is PoC-Near-Perfect with threshold $3/2$ iff G is (P_6, C_6) -free.

Theorem: $t = 2$ [Camby and Schaudt '14]

For every (P_8, C_8) -free graph G , $\gamma_c(G) \leq 2\gamma(G)$.

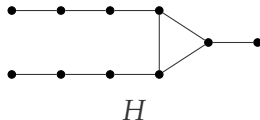
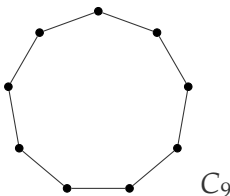
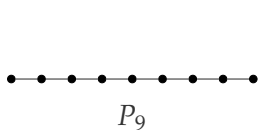
PoC-Near-Perfect graphs (with threshold 2)

Theorem: $t = 2$ [Camby and Schaudt '14]

For every (P_8, C_8) -free graph G , $\gamma_c(G) \leq 2\gamma(G)$.

Conjecture: [Camby and Schaudt '14]

For every graph G , G is PoC-Near-Perfect with threshold 2 iff G is (P_9, C_9, H) -free.



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- G : connected (P_9, C_9, H) -free graph s.t. $\gamma_c(G) \geq 2\gamma(G) + 1$;
- D^0 : $|D^0| = \gamma(G)$ and D^0 minimizes the number of cc;
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 $\Rightarrow D_1^0, \dots, D_g^0$ the cc of $G[D^0]$
- ℓ the largest integer s.t. there exists a dominating set D s.t.
 - $G[D]$ has $g - \ell$ cc;
 - $|D| \leq \gamma(G) + \ell$;
 - with some notion of heredity.

Sketch of the proof

$D_1 \cdots D_{g-\ell}$: the cc of $G[D]$.

Observation

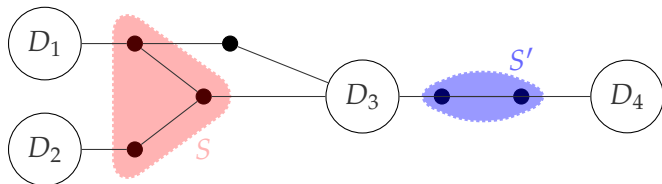
For every i , $\text{dist}(D_i, D \setminus D_i) = 3$.

Definition

Semi-relevant set $S \subseteq V \setminus D$ s.t. $G[D \cup S]$ has $g - \ell - |S| + 1$ cc.

\Rightarrow every semi-relevant set is connected.

Sketch of the proof



- $A(S) = \{D_1, D_2, D_3\}$;
- v_i^S : the vertex of S adjacent to someone in D_i ;
- there is a bijection between S and $A(S)$;
- for every v_i^S , there is $u_i \in N[D_i]$ with no neighbor in S ...

$\Rightarrow \nexists$ semi-relevant set S with $A(S) = \cup_i D_i$ (consider $D \cup S$).

Lemma

For every **maximal semi-relevant set** S , there is a semi-relevant set S' such that $A(S) \cup A(S') = \cup_i D_i$ and $|A(S) \cap A(S')| = 1$

- S' : semi-relevant set of maximal size built greedily...
- D_1 : the cc that belongs to $A(S)$ and $A(S')$.

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- D_1 : the cc that belongs to $A(S)$ and $A(S')$.
- $v_1^S v_1^{S'}$ is an edge (otherwise P_9 ☺).
- D_r : a cc. with no neighbor in $S \cup S'$.
- $\text{dist}(D_p, D_r) = 3$ for some $D_p \in A(S) \cup A(S')$.

Sketch of the proof

- $\gamma_c(G) = 2\gamma(G) + 1$ (consider $D \cup S \cup S'$);
 - $|D| = \gamma(G) + \ell$;
 - $|D_i^0| = 1$ for every $1 \leq i \leq \gamma(G)$;

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- If $D' = (D \setminus \{w_1\}) \cup S \cup S'$ is a connected dominating set:
 $|D'| = 2\gamma(G)$ ☺
- If not, w_1 has a private neighbor in $D' \Rightarrow H$ or P_9 ☺

Thank you for your attention!

Some extra...

n	4	5	6	7	8	9	10	11	12	13	14	15
$\gamma(P_n)$	2	2	2	3	3	3	4	4	4	5	5	5
$\gamma_c(P_n)$	2	3	4	5	6	7	8	9	10	11	12	13
PoC	1	$\frac{3}{2}$	2	< 2	2	$\frac{7}{3}$	2	$< \frac{7}{3}$	2.5	2.2	2.4	2.6