Shortest Cycle Cover

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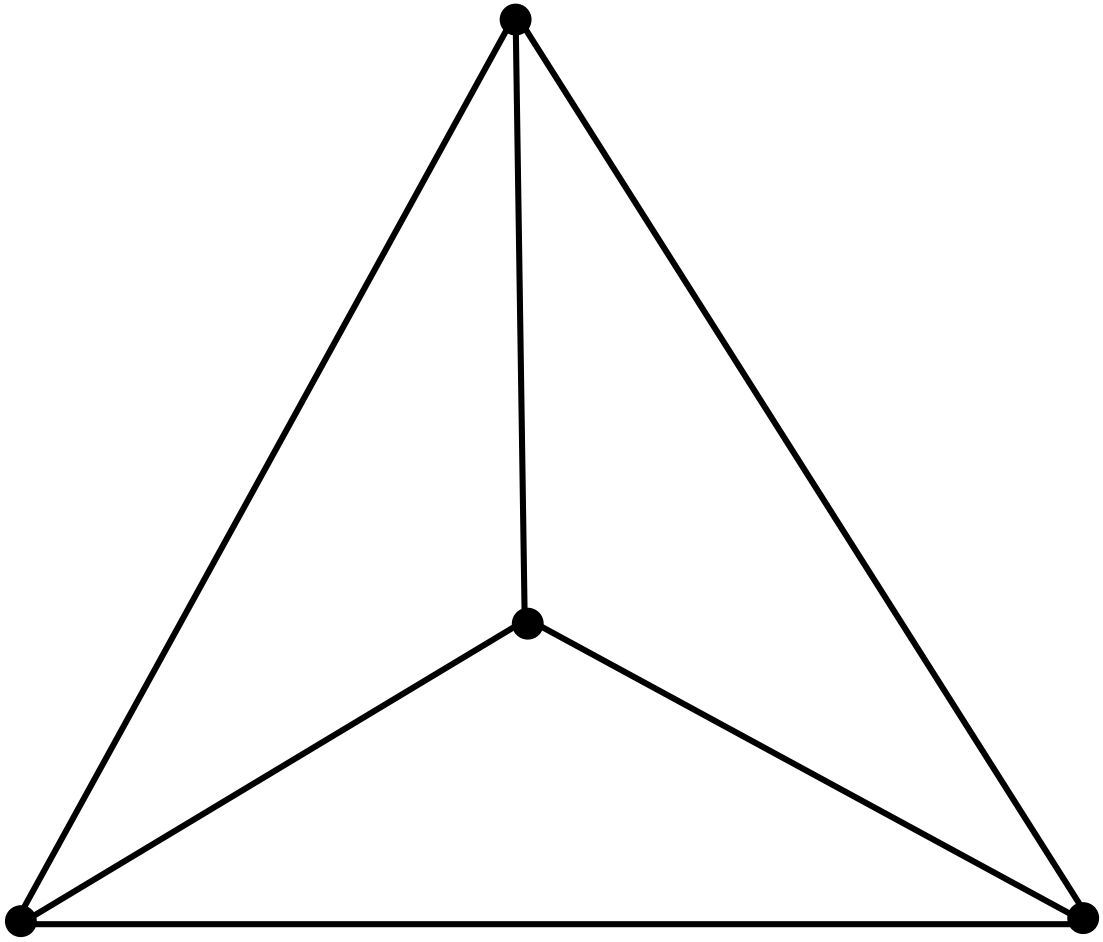
Circuit, Cycle

- A *circuit* is a connected 2-regular subgraph and a *cycle* is a union of edge-disjoint circuits. A *cycle cover* of a graph G is a family of cycles of G such that each edge of G is contained in at least one cycle of the family.
- The *length* of a cycle cover $\mathcal{C}$, denoted by $l(\mathcal{C})$, is the sum of the lengths (number of edges) of all the cycles in $\mathcal{C}$.

Example



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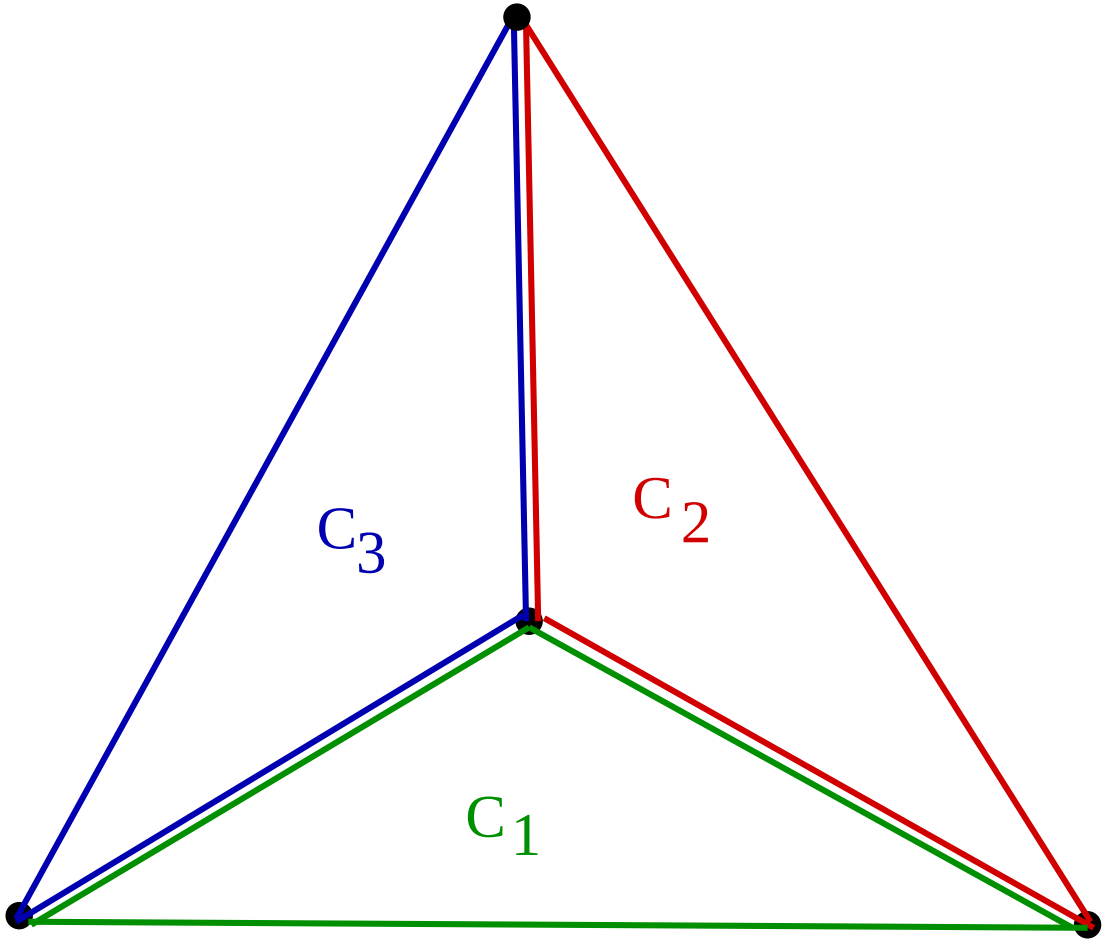
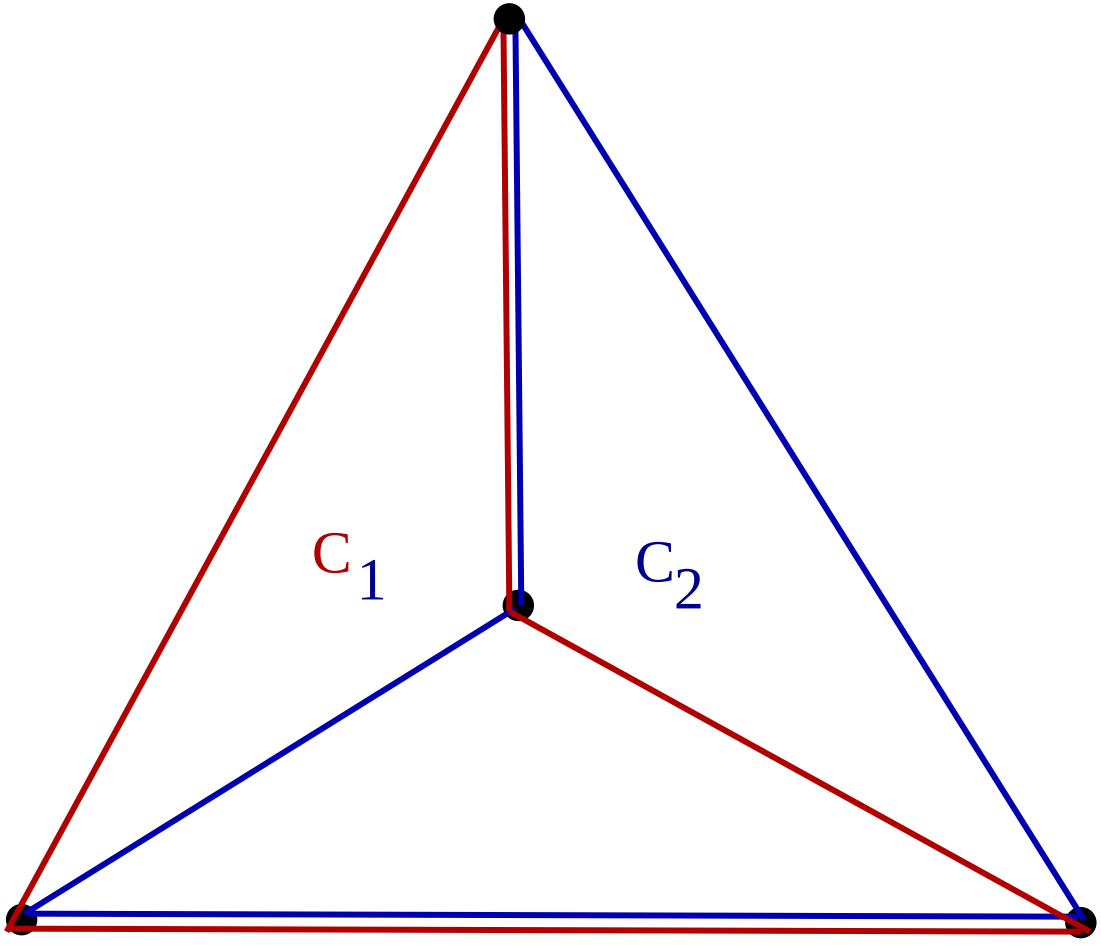


$$I(\mathcal{C}) = 9$$

Example



$$I(\mathcal{C}) = 8$$

Shortest Cycle Cover Problem (SCCP)

- Given a graph G and an integer k, does G have a cycle cover of total length at most k?
- Thomassen ('92) proved that the SCCP is a NP-complete problem

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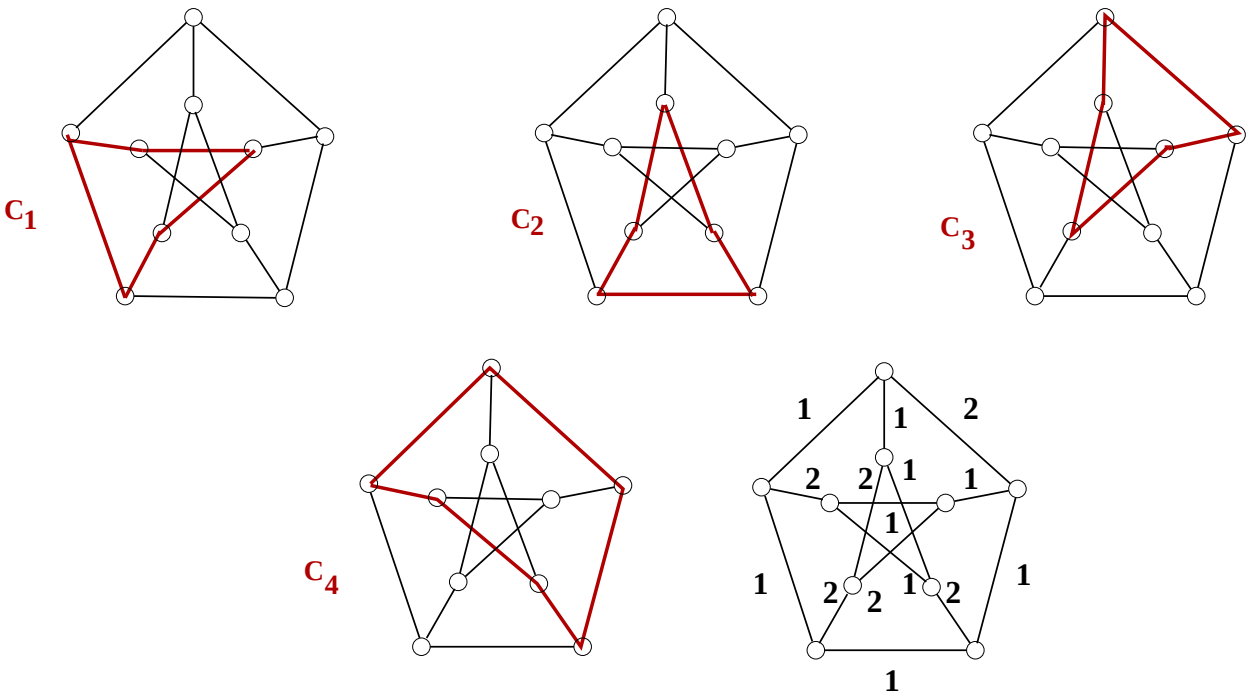
Conjecture

Every bridgeless graph $G = (V, E)$ has a cycle cover $\mathcal{C}$ such that:

$$I(\mathcal{C}) \leq \frac{7}{5}|E|.$$



$$I(\mathcal{C}) = 21$$

Cycle Double Cover

A *cycle double cover* **CDC** of a graph $G = (V, E)$ is a family of cycles of G such that every edge appears in exactly two cycles of this family.
The following double cover conjecture was proposed by several authors.

Conjecture (Szekeres '73, Seymour '79)

Every bridgeless graph has a cycle double cover.

Jamshi and Tarsi proved ('92):

Theorem

If the Conjecture concerning the existence of a cycle cover $\mathcal{C}$ of length at most $\frac{7}{5}|E|$ is true then the Cycle double cover conjecture is true.

First bound : Itai, Rodeh '78:

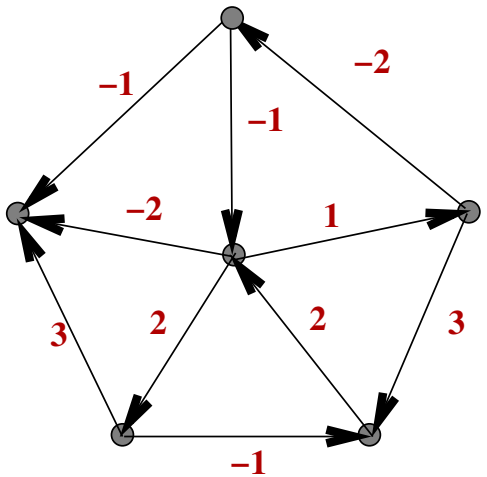
Theorem

Every bridgeless graph $G = (V, E)$ has a cycle cover $\mathcal{C}$ such that:

$$l(\mathcal{C}) \leq |E| + 2|V| log|V|.$$

Definition

Let M be an abelian group, $\vec{G}$ be an oriented graph,
$\phi : E(\vec{G}) \to M$ is a *flow* or an *M-flow* if :
$\sum_{e \in \Delta^+(v)} \phi(e) = \sum_{e \in \Delta^-(v)} \phi(e)$ for every vertex $v \in V(\vec{G})$.



If $M = \mathbb{Z}_2^k$ the condition is $\sum_{e \in \Delta(v)} \phi(e) = 0$

Definition

- The *support* $\sigma(\phi)$ of a flow ϕ in a digraph $G = (V, E)$ is the set of arcs e such that $\phi(e) \neq 0$.
- If $\sigma(\phi) = E$, ϕ is a M-nowhere zero flow (M-NZF).
- A $\mathbb{Z}$-flow is a k-flow if $\phi(E) \subset [-(k-1), 0[\cup]0, k-1]$
- $B \subseteq M$, $-B = B$
  ϕ is a *B-flow*,
  if $\phi : E \to M$ is a flow s.t. $\phi(E) \subseteq B$

If $M = \mathbb{Z}_2$, $\sigma(\phi)$ is a cycle.

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Theorem (Tutte '50)

Let G be a digraph. For every $k \geq 2$, the following conditions are equivalent:

1. *There exists a $\mathbb{Z}_k$-NZF in G.*
2. *For any Abelian group M of order k, there exits a M-NZF in G.*
3. *There exists a k-NZF in G.*

Tutte's Conjectures

Conjecture (Tutte '54)

Every bridgeless graph without 3-cuts has a 3-NZF.

Theorem (Jaeger'79)

Every bridgeless graph without 3-cuts has a 4-NZF.

Conjecture (Tutte '54)

Every bridgeless graph has a 5-NZF.

Theorem

Every bridgeless graph has:

1. *a 8-NZF (Jaeger '79)*
2. *a 6-NZF (Seymour '81)*

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Conjecture (Peissmann '81, Celmins '84)

- *For every graph with no cut-edge, there is a list of five cycles so that every edge is contained in exactly two.*
- *Every graph with no cut-edge has a B-flow for the set $B \subseteq \mathbb{Z}_2^5$ consisting of those vectors with exactly two 1's.*

Conjecture (Jaeger '88)

- *For every oriented graph with no cut-edge, there is a list of five 2-flows $\phi_1, \phi_2, \ldots, \phi_5$ with $\sum_{i=1}^{5} \phi_i = 0$ such that every edge is in the support of exactly two of these flows.*
- *Every graph with no cut-edge has a B-flow for the set $B \subseteq \mathbb{Z}^5$ consisting of those vectors with exactly three 0's, one 1, and one -1.*

Conjecture (Fulkerson,'71)

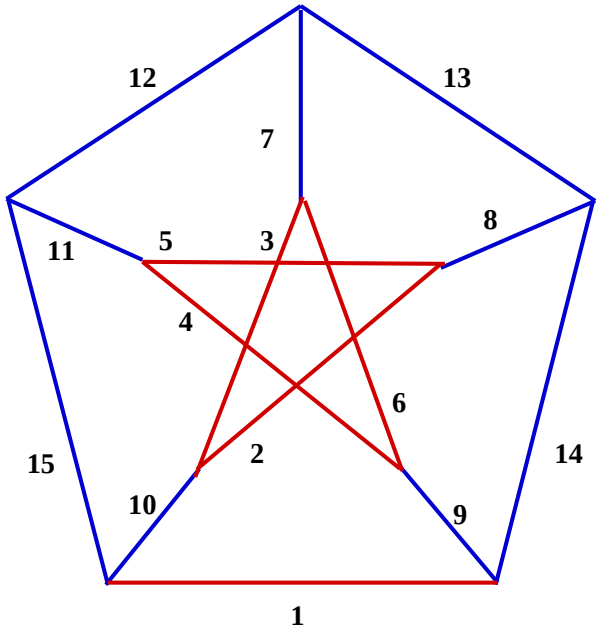
- *For every cubic graph with no cut-edge, there is a list of 6 perfect matchings so that every edge is contained in exactly two.*
- *Every graph with no cut-edge has a B-flow for the set $B \subseteq \mathbb{Z}_2^6$ consisting of those vectors with exactly four 1's.*

Conjecture (Berge,'79)

The edges of any cubic graph with no cut-edge can be covered by 5 perfect matchings.

Petersen flow

	C_1	C_2	C_3	C_4	C_5	C_6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	0	1	0	0	0
4	0	0	0	1	0	0
5	0	0	0	0	1	0
6	0	0	0	0	0	1
7	0	0	1	0	0	1
8	0	1	0	0	1	0
9	0	0	0	1	0	1
10	0	1	1	0	0	0
11	0	0	0	1	1	0
12	1	1	1	1	1	0
13	1	1	0	1	1	1
14	1	0	0	1	0	1
15	1	1	1	0	0	0



Conjecture (Jaeger)

Every bridgeless graph has a Petersen flow.

Theorem (Bermond, Jackson, Jaeger)

If $G = (V, E)$ has a 4-NZF then it has a cycle cover $\mathcal{C}$ such that:

$$I(\mathcal{C}) \leq \frac{4}{3}|E|$$

Proof: 4-NZF $\Longleftrightarrow$ $\mathbb{Z}_2 \times \mathbb{Z}_2$-NZF.
Φ, ϕ_1, ϕ_2 *s.t.* $\Phi(e) = (\phi_1(e), \phi_2(e))$, ϕ_i *is a* $\mathbb{Z}_2$*-flow,*
$E = \sigma(\phi_1) \cup \sigma(\phi_2)$.
$C_1 = \sigma(\phi_1)$, $C_2 = \sigma(\phi_2)$, $C_3 = C_1 \triangle C_2$

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4-NZF and Cycle cover

C_1	C_2	C_3
0	1	1
1	0	1
1	1	0

$$|C_1| + |C_2| + |C_3| = 2|E|$$

$\exists\, k \in \{1,2,3\}$ s.t. $|C_k| \geq \frac{2}{3}|E|$

$\{C_i, C_j\}$ $(i \neq j)$ is a cycle cover

Let $\mathcal{C} = \{C_i, C_j\}$, $i \neq j \neq k$:

$$I(\mathcal{C}) \leq 2|E| - \frac{2}{3}|E|$$

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Results

$c(G)$ the size od a smallest cycle cover of a bridgeless graph G.

Authors	Results
Itai, Rodeh ('78)	$c(G) \leq \|E\| + 2\|V\|log\|V\|$.
Bermond, Jackson, Jaeger ('83) Alon, Tarsi ('85)	$c(G) \leq \frac{5}{3}\|E\|$. $c(G) \leq \frac{5}{3}\|E\|$.
Jamshy, R. , Tarsi ('87)	If G has a 5-NZF $c_3(G) \leq \frac{8}{5}\|E\|$.
Jackson ('90)	If G is $2k$-edge-connected ($k \geq 2$), then $c(G) \leq \frac{2k+2}{2k+1}\|E\|$.

Results

Authors	Results
R. (’91)	If G has a 4-NZF $G \neq K_4$, then $c_2(G) \leq \lvert E \rvert + \lvert V \rvert - 3$.
Fan, R. (’92)	If Fulkerson conjecture is true $c_3(G) \leq \frac{22}{15}\lvert E \rvert$.
R. (’92)	If Petersen conjecture is true $c_4(G) \leq \frac{7}{5}\lvert E \rvert$.
Fan (’93)	If 3-flow conjecture of Tutte is true $c_3(G) \leq \frac{7}{5}\lvert E \rvert$.
Fan (’98)	$c(G) \leq \lvert E \rvert + \lvert V \rvert - 1$.
Shu, Zhang (’04)	G with odd edge-connectivity $r > 3$ $c_3(G) \leq \frac{r+1}{r}\lvert E \rvert$.

Circular flow

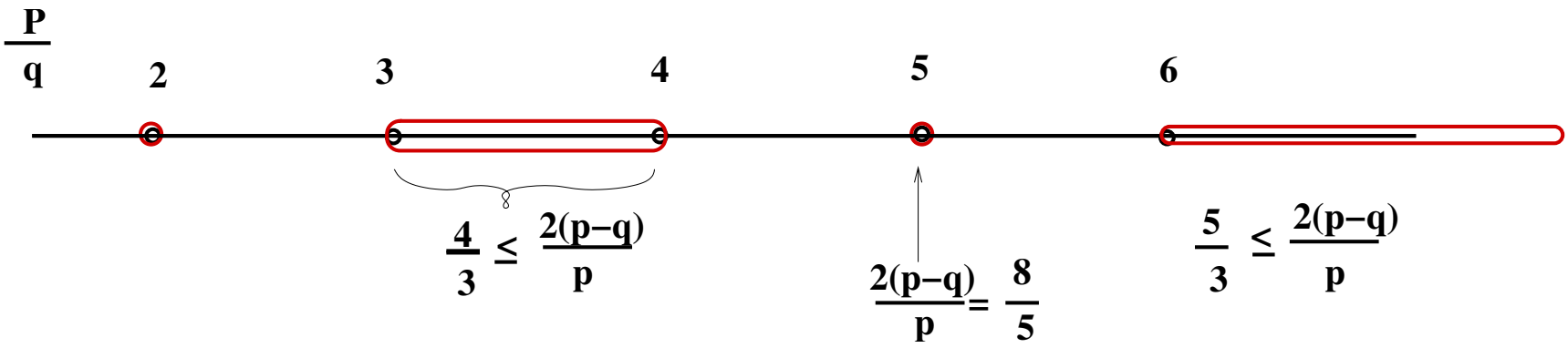
Let $G = (V, E)$ be a directed graph. $p \geq 2q$ integers. A flow $f : E \longrightarrow \{-(p-1), \ldots, p-1\}$ is a (p, q)-flow if for every edge $e \in E$ $q \leq |f(e)| \leq p - q$.

A $(k, 1)$-flow is a k-NZF

Conjecture (R., Zhu '05)

If $G = (V, E)$ has a (p, q)-flow then it has a cycle cover $\mathcal{C}$ with

$$l(\mathcal{C}) \leq \frac{2(p-q)}{p}|E|$$



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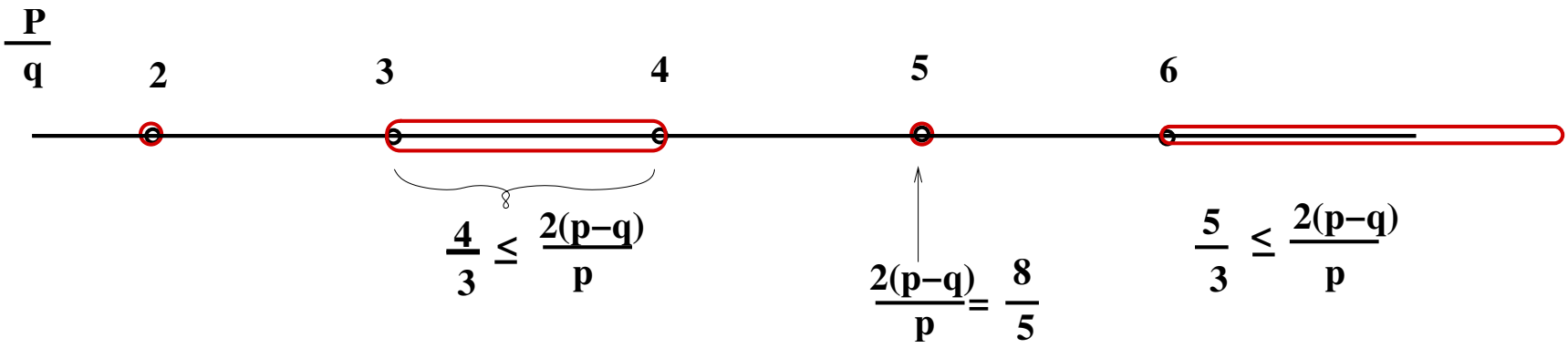
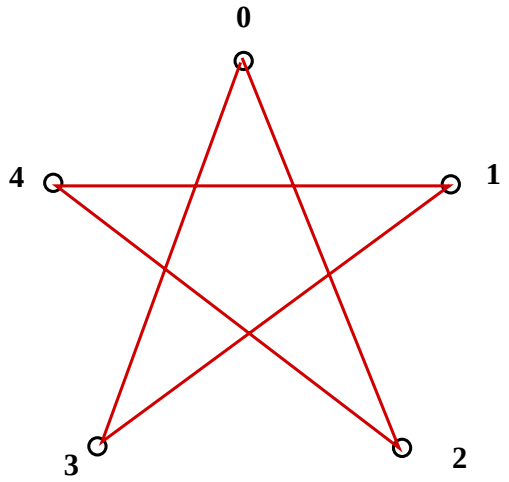


Circular coloring

Let $G = (V, E)$ be a graph. $p \geq 2q$ integers. $G = (V, E)$ is (p, q)-colorable if there exists a mapping $f : V \longrightarrow \{0, \ldots, p-1\}$ s.t. for every edge $xy \in E$ (Vince '88):

$$p \leq |f(x) - f(y)| \leq p - q$$

$$\chi_c(G) = \inf\{\frac{p}{q} : \text{ there is a } (p, q) \text{ -coloring of } G\}$$



$(5, 2)$-coloring of C_5

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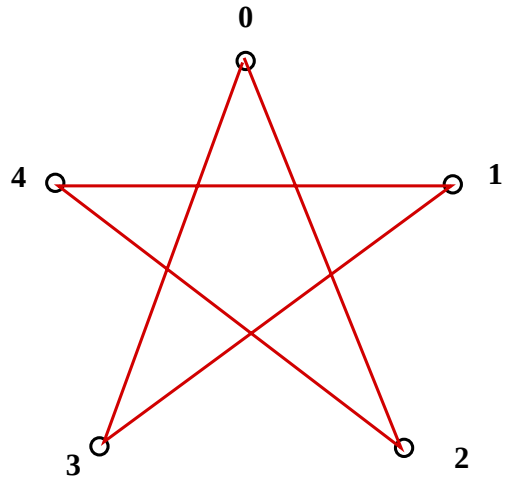
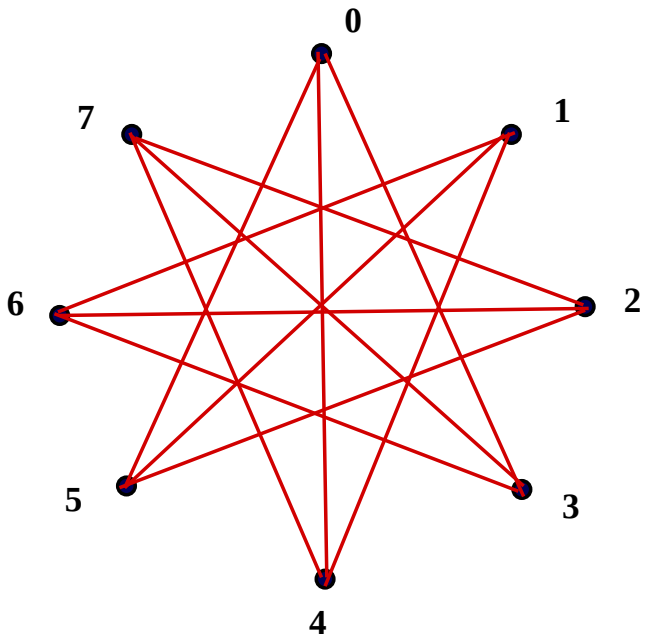


$(5, 2)$-coloring of C_5

Circular clique

$K_{\frac{p}{q}}$ be the graph with vertex set $\{0, \ldots, p-1\}$ ($p \geq 2q$) in which ij is an edge if and only if $q \leq |j-i| \leq p-q$.



Circular clique $K_{\frac{8}{3}}$

Bicircular flow

- A CDC $\mathcal{C} = \{C_0, \ldots, C_{p-1}\}$ is said to be (p, q)-colorable if the intersection graph $G_{\mathcal{C}}$ is (p, q)-colorable.
- $V(G_{\mathcal{C}}) = \{0, \ldots, p-1\}$ and $ij \in E(\mathcal{C})$ iff $C_i \cap C_j \neq \emptyset$.
- $D_{p,q}$ is the subset of $\mathbb{Z}_2^p$ consisting of elements $(z_0, \ldots, z_{p-1})$ with exactly two elements z_i,z_j equal to one and all the other equal to 0, with $p \leq |j - i| \leq p - q$.

Proposition

G has a $D_{p,q}$-flow iff G has a (p, q)-colorable CDC

Bicircular flow

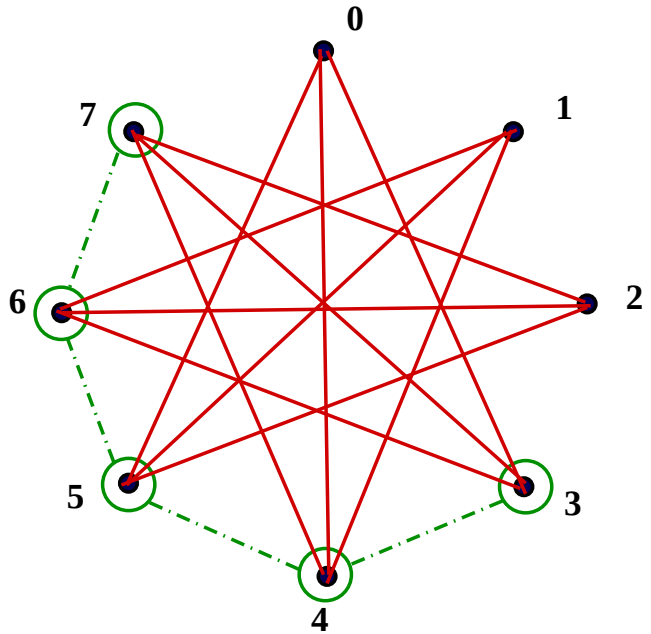
Theorem (R., Zhu '05)

If G has a $D_{p,q}$-flow then it has a cycle cover $\mathcal{C}$ with

$$l(\mathcal{C}) \leq \frac{2(p-q)}{p}|E|$$

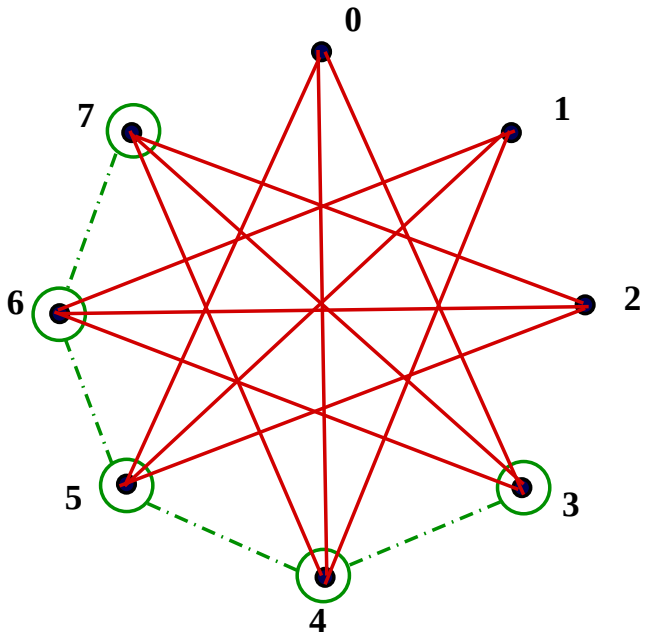
Proof

- G has a $D_{p,q}$-flow $\iff$ $G_{\mathcal{C}}$ the intersection graph of the corresponding CDC is a sub-graph of $K_{\frac{p}{q}}$
- $e_{i,j} = \{e \in E : e \in C_i \cap C_j\}$.
- $e \in e_{i,j} \iff e$ is of type ij
- there is a bijection between the $e_{i,j}$'s and the edges of $K_{\frac{p}{q}}$



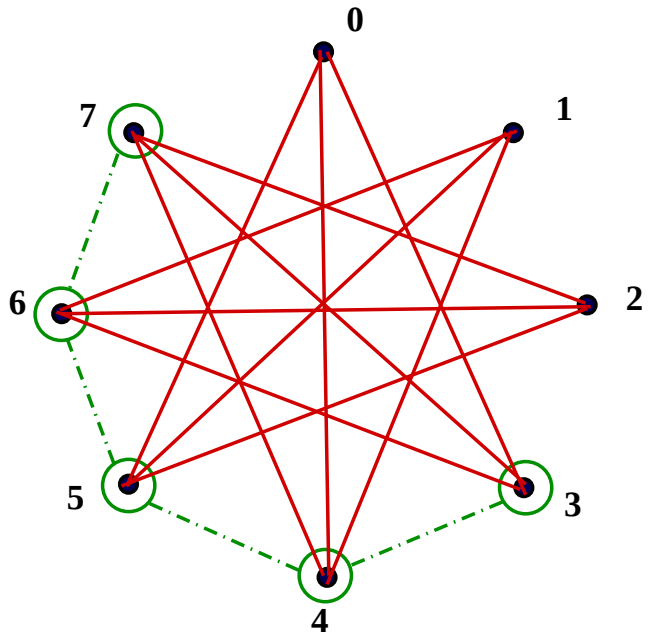
Proof $(8,3)$-flow

- $C_{012} = \{C_3, C_4, C_5, C_6, C_7\}$ is a cycle cover of the graph
- The edges of types $e_{i,j}$ $i \in \{0,1,2\}$, $j \in \{3,4,5,6,7\}$ are covered once by this cycle cover C_{012}.
- The edges of the remaining types $e_{3,7}$,$e_{3,6}$,$e_{4,7}$ are covered twice.



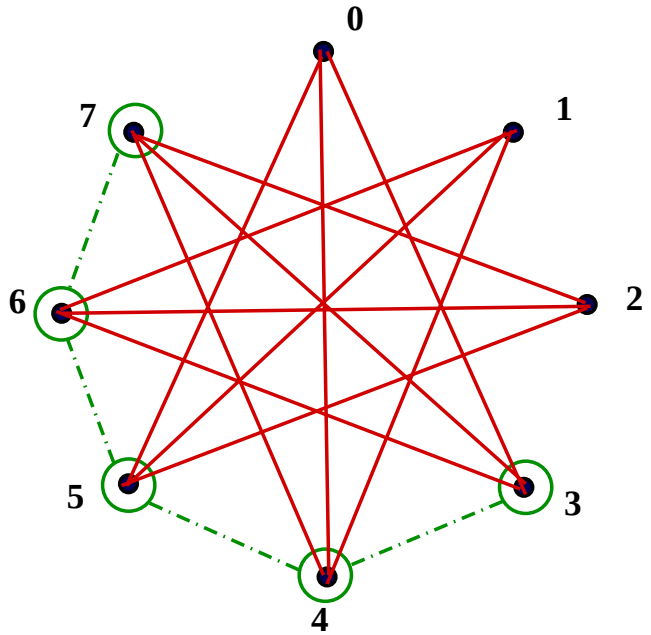
Proof $(8,3)$-flow

- $I(C_{012}) = |E| + |e_{3,7}| + |e_{3,6}| + |e_{4,7}|$
- $I(C_{012}) = |E| + |R_O|$ (R_0 the edges of the remaining types)
- $K_{\frac{p}{q}}$ is vertex transitive. C_{012} is denoted by $\mathcal{C}_0$.
- $\mathcal{C}_i$, $i \in \{0, \dots 7\}$ are 8 cycle covers (p)
- $I(\mathcal{C}_i) = |E| + |R_i|$



Proof $(8, 3)$-flow

- Each R_i contains 3 types of edges ($\frac{(p-2q+1)(p-2q)}{2}$)
- $\sum_0^7 I(\mathcal{C}_i) = 8|E| + \sum_0^7 |R_i|$
- $\sum_0^{p-1} I(\mathcal{C}_i) = p|E| + \sum_0^{p-1} |R_i|$
- each edges of types belonging to the reamining edges are counted twice ($p - 2q$ times)
- $\sum_0^7 I(\mathcal{C}_i) = 8|E| + 2|E|$
- $\sum_0^{p-1} I(\mathcal{C}_i) = p|E| + (p - 2q)|E|$



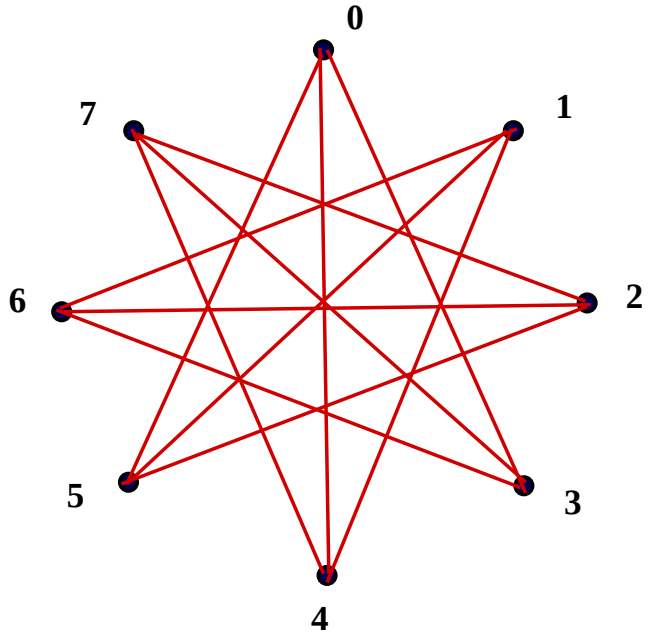
Proof $(8,3)$-flow

- There is $i \in \{0,\ldots,7\}$ ($i \in \{0,\ldots,p-1\}$) such that

$$I(\mathcal{C}_i) \leq \frac{5}{4}|E|$$

-

$$I(\mathcal{C}_i) \leq \frac{2(p-q)}{p}|E|$$

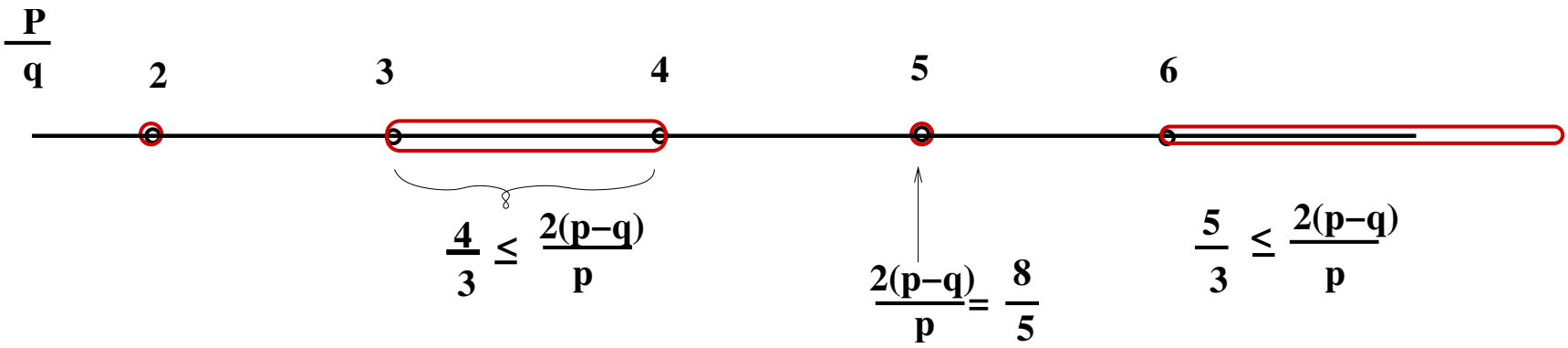


Planar graphs

Corollary

If a planar graph G has a (p,q)-flow then it has a cycle cover $\mathcal{C}$ with

$$l(\mathcal{C}) \leq \frac{2(p-q)}{p}|E|$$



Bicircular flows

Theorem (Jaeger '81)

If G has a $(2k+1, k)$-flow then it has a $D_{2k+1,k}$-flow

Corollary

If G has a $(2k+1, k)$-flow then it has a cycle cover $\mathcal{C}$ with

$$l(\mathcal{C}) \leq \frac{2k+2}{2k+1}|E|$$