# Homomorphisms of sparse graphs to small graphs

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## Flow

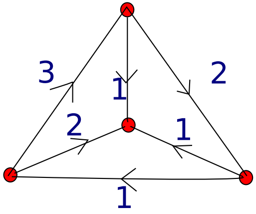
Let $G$ be a directed graph, $A$ be an abelian group.

## $A$-flow

A-flow $f$ of $G$ is a mapping which assigns to each edge $e$ of $G$ an element of $A$ $f(e)$ such that :

- for every vertex $x$, $\Sigma_{e \in E^+(x)} f(e) = \Sigma_{e \in E^-(x)} f(e)$.

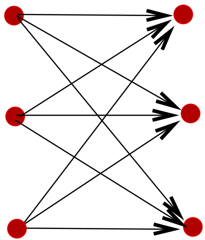
Here $E^+(x)$ is the set of edges incident to and oriented away from $x$, and $E^-(x)$ is the set of edges incident to and oriented towards $x$.




$\mathbb{Z}_4$-flow

# Modular orientation

- A graph is mod $(2p+1)$-orientable $(p \geq 1)$ if it has an orienation s.t. the out degree of each vertex is congruent mod $(2p+1)$ to the in-degree.
- $U(\mathbb{Z}_{2p+1}) = \{1, -1\}$



mod (3)-orientation of $K_{3,3}$

# Circular Flow

Let $G$ be a directed graph. For positive integers $p \geq 2q$, a $(p,q)$-flow $f$ of $G$ is a mapping which assigns to each edge $e$ of $D$ an integer $f(e)$ such that

- for every vertex $x$, $\Sigma_{e \in E^+(x)} f(e) - \Sigma_{e \in E^-(x)} f(e) = 0$,
- for every edge $e$, $q \leq |f(e)| \leq p - q$.

Here $E^+(x)$ is the set of edges incident to and oriented away from $x$, and $E^-(x)$ is the set of edges incident to and oriented towards $x$.

# JAEGER

**Theorem (Jaeger '82)**

*For any graph $G$ and for any $p \geq 1$ the following properties are equivalent:*

1. $G$ *is* $(2p+1)$*-orientable*
2. $G$ *has a* $U(\mathbb{Z}_{2p+1})$*-flow*
3. $G$ *has a* $(2p+1,p)$*-flow*

**Conjecture (The circular flow conjecture, Jaeger '82)**

*For all $p \geq 1$, every $4p$-edge connected graph has a $(2p+1,p)$-flow.*

# JAEGER

**Conjecture (The circular flow conjecture, Jaeger '82)**

*For all $p \geq 1$, every $4p$-edge connected graph has a $(2p+1, p)$-flow.*

- $p = 1 \Longrightarrow$ 3-flow conjecture of Tutte ('54)
- $p = 2 \Longrightarrow$ 5-flow conjecture of Tutte ('54)

# Circular coloring

Let $G$ be a graph.
For positive integers $p \geq 2q$, a $(p,q)$-coloring of $G$ is a mapping $f : V(G) \longrightarrow \{0, 1, \cdots, p-1\}$ such that

- for every edge $xy$, $q \leq |f(x) - f(y)| \leq p - q$

**Circular chromatic number**

$$\chi_c(G) = min\{\frac{p}{q} : \ G \text{ has a } (p,q)\text{-coloring}\}$$

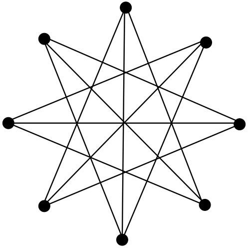
Jeager's conjecture restricted to planar graphs

**Conjecture**

*Every planar graph $G$ of girth at least $4p$ has circular chromatique number at most $2 + \frac{1}{p}$*

# Circular clique

Let $K_{\frac{p}{q}}$ be the graph with vertex set $\{0,\ldots,p-1\}$ $(p \geq 2q)$ in which $ij$ is an edge if and only if $q \leq |j-i| \leq p-q$. Such a graph is called a circular clique.



$K_{\frac{8}{3}}$

# Homomorphism

## Homomorphism

A *homomorphism* from $G$ to $H$ is a mapping $h : V(G) \rightarrow V(H)$ such that if $xy \in E(G)$ then $h(x)h(y) \in E(H)$.

## Circular coloring-Homomorpism

$G$ has a $(p, q)$-coloring $\Longleftrightarrow$ it exist a homomorphism from $G$ to $K_{\frac{p}{q}}$ .

Denoted by

$$G \longrightarrow K_{\frac{p}{q}}$$

.

# Homomorphism

Let $g(G)$ be the girth of $G$

**Conjecture (Jaeger)**

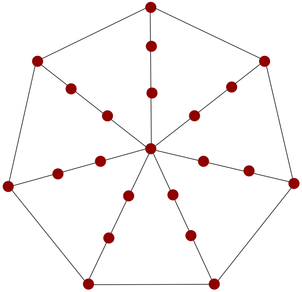
*Let $G$ be a planar graph:*

$$g(G) \geq 4p \Longrightarrow G \to K_{\frac{2p+1}{p}}$$

Remark: $K_{\frac{2p+1}{p}} = C_{2p+1}$

$4p$ is best possible.

$p = 2$



DeVos, Pirnazar and Ullman

# Results

Let $G$ be a planar graph.
We denote by $odg(G)$ the odd girth of a graph $G$

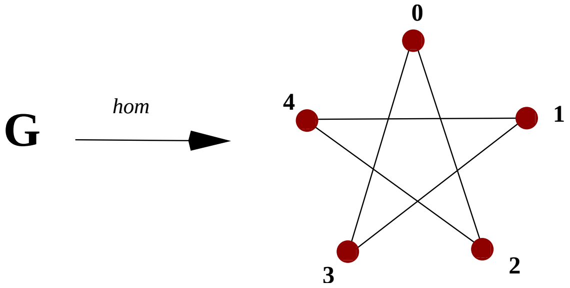
**Theorem**

$\chi_c(G) \leq 2 + \frac{1}{p}$ *if:*

- $g(G) \geq 10p - 4$: *Nešetřil and Zhu ('96), Gallucio, Goddyn, Hell ('01)*
- $odg(G) \geq 10p - 3$: *Klostermeyer and Zhang ('00)*
- $odg(G) \geq 8p - 3$: *Zhu ('01)*
- $g(G) \geq \frac{20p-2}{3}$: *Borodin, Kim, Kostochka, West ('04)*

# Results

Let $G$ be a planar graph, if $odg(G) \geq 13$ ($p = 2$), by the two last results we have:



# Conjecture

**Conjecture (Jaeger '82)**

*Let $G$ be a planar graph, if $g(G) \geq 4p$ then $\chi_c(G) \leq 2 + \frac{1}{p}$*

**Conjecture (Klostermeyer and Zhang ('00))**

*Let $G$ be a planar graph, if $odg(G) \geq 4p + 1$ then $\chi_c(G) \leq 2 + \frac{1}{p}$*

# Maximum average degree

**Definition-Maximum average degree**

$$\mathrm{Mad}(G) = max\left\{\frac{2 \cdot |E(H)|}{|V(H)|}, H \subseteq G\right\}.$$

if $G$ is a planar graph with girth $g$, then $\mathrm{Mad}(G) < \frac{2g}{g-2}$.

The maximum average degree of a graph can be computed in polynomial time by using the matroid partitoning algorithm EDMONDS '65.

# Results

**Theorem ( Borodin, Kim, Kostochka, West ('04))**

*If* $g(G) \geq 6p - 2$ *and* $Mad(G) < \frac{10p-1}{5p-2}$ *then* $\chi_c(G) \leq 2 + \frac{1}{p}$

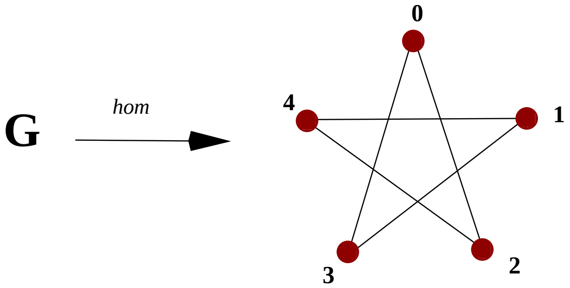
**Theorem ( Borodin, Hartke, Ivanova, Kostochka, West ('04))**

*Let* $G$ *be a triangle-free graph. If* $Mad(G) < \frac{12}{5}$ *then* $\chi_c(G) \leq \frac{5}{2}$

# Results

## Corollary

*Let $G$ be planar graph. If $g(G) \geq 12$ then $\chi_c(G) \leq \frac{5}{2}$*



# Results

**Theorem (R., Roussel ('05))**

*Let $G$ be a triangle free graph:*

- *if* $Mad(G) < \frac{22}{9}$ *then* $\chi_c(G) \leq \frac{11}{4}$
- *if* $Mad(G) < \frac{5}{2}$ *then* $\chi_c(G) \leq \frac{14}{5}$

# Results

**Corollary**

*Let $G$ be a triangle free planar graph:*

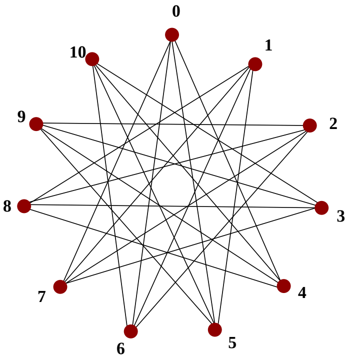
- *if* $g(G) \geq 11$ *then* $\chi_c(G) \leq \frac{11}{4}$
- *if* $g(G) \geq 10$ *then* $\chi_c(G) \leq \frac{14}{5}$

**Conjecture (R., Roussel ('09))**

*Let $G$ be with* $g(G) \geq 4p$ *and* $Mad(G) < 2 + \frac{2}{2p-1}$ *then* $\chi_c(G) \leq 2 + \frac{1}{p}$

# Results

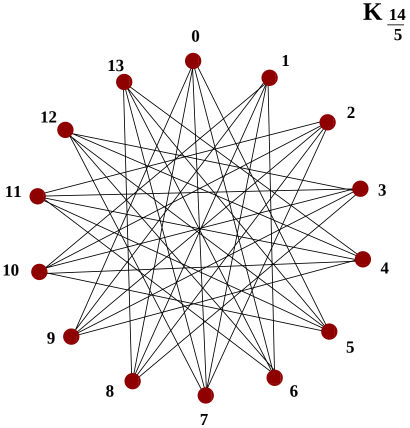
if $\mathrm{Mad}(G) < \frac{22}{9}$ then $G \longrightarrow$




$\mathbf{K}_{\frac{11}{4}}$

# Results

if $\mathrm{Mad}(G) < \frac{5}{2}$ then $G \longrightarrow$



# Fractional coloring

Let $a > b$ be two integers. We denote by $\mathcal{P}_b[a]$ the b-elements subsets of a the set $\{1, \cdots, a\}$. Let $G$ be a graph and $c$: $V(G) \rightarrow \mathcal{P}_b[a]$ such that for two adjacent vertices $u$ and $v$ of $G$ $c(u)$ and $c(v)$ are disjoint. $c$ is then an $(a, b)$-coloring.

## Fractional chromatic number

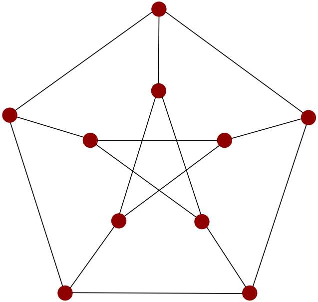
$$\chi_f(G) = min\{\frac{a}{b} : \; G \text{ has a } (a, b)\text{-coloring}\}$$

$$\boxed{\chi_f(G) \leq \chi_c(G)}$$

# Kneser graphs

## Kneser graph

The *Kneser graph*, denoted by $K_{n:k}$, is defined to be the graph in which vertices represent subsets of cardinality $k$ taken from $\{1, 2, \cdots, n\}$ and two vertices are adjacent if and only if the corresponding subsets are disjoint.



$K_{5:2}$

# Homomorphism

**Fractional coloring-Homomorphism**

$G$ has a $(a, b)$-coloring $\Longleftrightarrow$ it exists a homomorphism from $G$ to $K_{a:b}$ .

Denoted by

$$G \longrightarrow K_{a:b}$$

.

# Homomorphism

Let $g(G)$ be the girth of $G$

**Conjecture**

*Let $G$ be a planar graph:*

$$g(G) \geq 4p \Longrightarrow G \to K_{2p+1:p}$$

Jaeger conjecture $\Longrightarrow$ the above conjecture.
Remark: $K_{\frac{2p+1}{p}} = 0_{2p+1}$ the Odd graph.

# Klostermeyer and Zhang ('00)

**Theorem (Klostermeyer and Zhang ('00))**

*Let $G$ be a planar graph.*
(1) *If the odd-girth of $G$ is at least $10p-7$ with $p \geq 2$, then $\chi_f(G) \leq 2+\frac{1}{p}$.*
(2) *If $g(G) \geq 10p-9$ with $p \geq 2$ and $\Delta(G) \leq 3$, then $\chi_f(G) \leq 2+\frac{1}{p}$.*
(3) *There exists a planar graph $H$ with odd-girth at least $2p+1$ such that $\chi_f(G) > 2+\frac{1}{p}$.*

# Results

**Theorem (Pirnazar and Ullman ('02))**

*Let $G$ be a planar graph. If $g(G) \geq 8p - 4$ ($p \geq 1$) then $\chi_f(G) \leq 2 + \frac{1}{p}$.*

**Theorem (Dvořák, Škrekovski and Valla ('08))**

*Let $G$ be a planar graph. If $odg(G) \geq 9$ then $\chi_f(G) \leq \frac{5}{2}$.*

**Conjecture (Naserasr ('09))**

*Let $G$ be a planar graph. If $odg(G) \geq 2p + 3$ ($p \geq 1$) then $\chi_f(G) \leq 2 + \frac{1}{p}$.*

# Maximum average degree

**Theorem (Chen, R. ('08))**

*Let $G$ is a triangle-free graph.*

- *If $Mad(G) < \frac{5}{2}$, then $\chi_f(G) \leq \frac{5}{2}$*
- *If $Mad(G) < \frac{9}{4}$, then $\chi_f(G) \leq \frac{7}{3}$*
- *If $Mad(G) < \frac{24}{11}$, then $\chi_f(G) \leq \frac{9}{4}$*

# Maximum average degree

**Corollary**

*Let $G$ is a planar graph.*

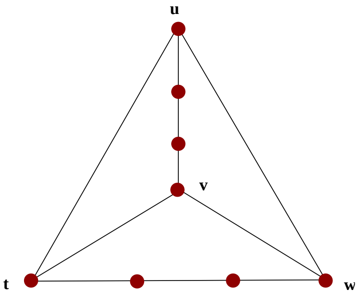
- *If $g(G) \geq 10$, then $\chi_f(G) \leq \frac{5}{2}$*
- *If $g(G) \geq 18$, then $\chi_f(G) \leq \frac{7}{3}$*
- *If $g(G) \geq 24$, then $\chi_f(G) \leq \frac{9}{4}$*

The Theorem of Dvořák, Škrekovski and Valla gives a better result than the first item of Corollary. They proved that if $odg(G) \geq 9$ then $\chi_f(G) \leq \frac{5}{2}$.

# Maximum average degree

- If $Mad(G) < \frac{5}{2}$, then $\chi_f(G) \leq \frac{5}{2}$

It is best possible.
Let $G_2$ be the following graph:

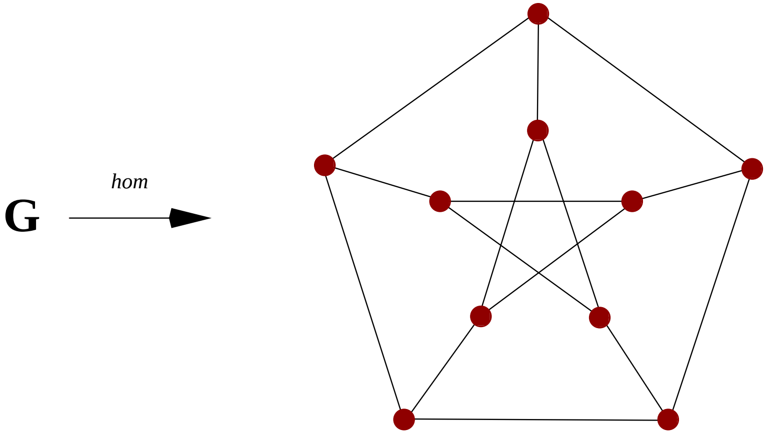



$$Mad(G_2) = \frac{5}{2}$$

$G_2$ does not have a $(5, 2)$- coloring.

# If $Mad(G) < \frac{5}{2}$, then $\chi_f(G) \leq \frac{5}{2}$

We want to prove that: If $G$ is triangle-free and $Mad(G) < \frac{5}{2}$, then



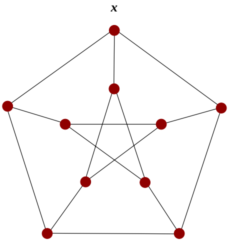
# sketch of the proof

**sketch**

We take a minimum counterexample. We prove that this minimum counterexample cannot contain some configurations. If a graph does not contain these configurations then $Mad(G) \geq \frac{5}{2}$, a contradiction.

# Observations

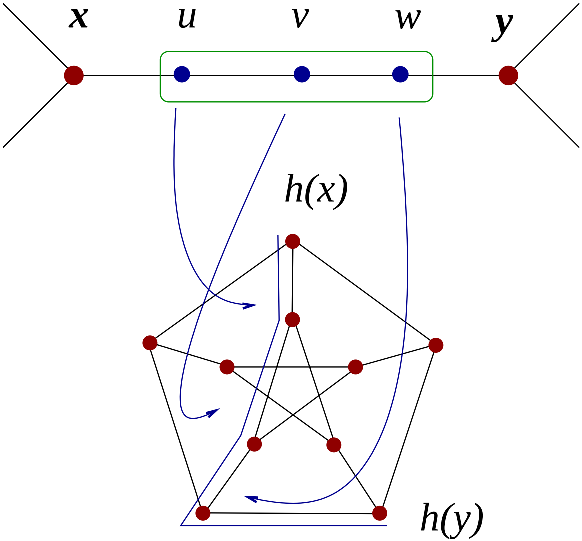
For $x \in V(K_{5:2})$, we define $L_i(x) = \{y|$ there is a walk of length $i$ in $K_{5:2}$ joining $x$ and $y\}$ and $F_i(x) = V(K_{5:2}) \backslash L_i(x)$.




## Observation

For $x \in V(K_{5:2})$, we have that $|F_1(x)| = 7$, $|F_2(x)| = 3$, $|F_3(x)| = 1$, and $|F_{4^+}(x)| = 0$.

# Forbidden configurations



# Forbidden configurations

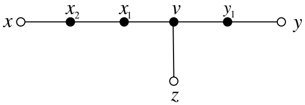



Fig.2: v is a (2,0,1)-vertex.

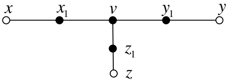



Fig.3: A $(1^+,1^+,1^+)$-vertex v.

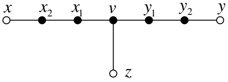



Fig.4: A $(2,2,0^+)$-vertex v.

# Forbidden configurations

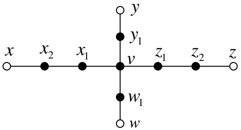



Fig.5: A $(1^+,1^+,2,2)$-vertex v.

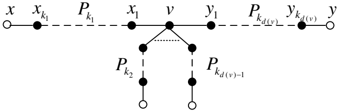



Fig.6: A united thread structure $P_{k_1}(k_1,\cdots,k_{d(v)})P_{k_{d(v)}}$ with a knot $v=(k_1,\cdots,k_{d(v)})$.

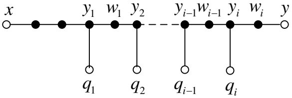



Fig.7: $P_2(2,0,1)P_1^i$.

# Forbidden configurations

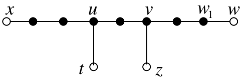



Fig.8: $P_2(2,0,1)P_1(1,0,2)P_2$.

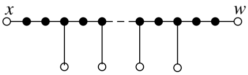



Fig.9: $P_2(2,0,1)P_1^i(1,0,2)P_2$.

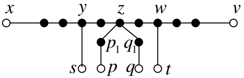



Fig.10: $P_2(2,0,1)P_1(1,1^+,1,1)P_1(1,0,2)P_2$

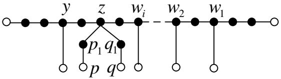



Fig.11: $P_2(2,0,1)P_1(1,1^+,1,1)P_1^i(1,0,2)P_2$

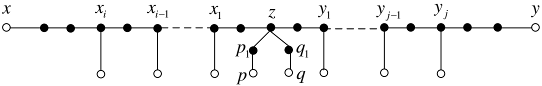
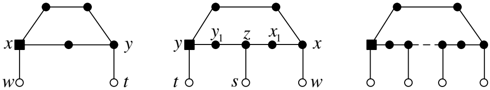


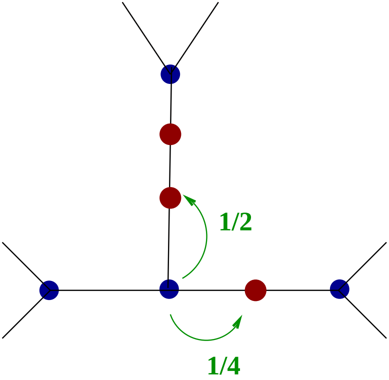

Fig.12: $P_2(2,0,1)P_1^i(1,1^+,1,1)P_1^j(1,0,2)P_2$




(1) (2) (3)

Fig.13: Reducible united thread-cycle structures in Claim 16.

# Observation

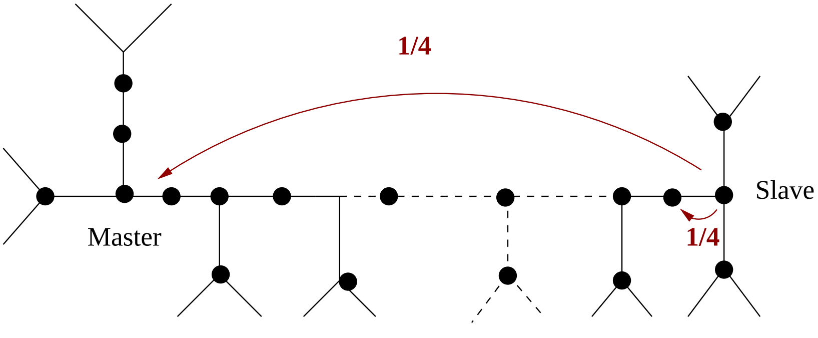


# Master-Slave

## Definition (Borodin, Hartke, Ivanova, Kostochka, West ('08))

A *compensatory path* for a $(2,0,1)$-vertex $v$ is chosen as any shortest path $F$ formed by concatenating threads in the following way. First, $F$ starts along the unique 1-thread at $v$. Then $F$ traversed some number of 1-threads by $(1,0,1)$-vertices. Let $v^*$ be the first vertex reached which is not a $(1,0,1)$-vertex. Moreover, we say that $v^*$ is a *slave* of $v$ and $v$ is a *master* of $v^*$.

# Observation




**Lemma**

*Suppose $v$ is a $(2,0,1)$-vertex. Let $v^*$ be the slave of $v$. Then the following hold:*
$(1)$ *$v^*$ is neither a 2-vertex nor a $(1,0,1)$-vertex;*
$(2)$ *If $d(v^*) = 3$ then $v^*$ is a $(1,0,0)$-vertex.*

# Discharging rules

Initial charge $\omega(v) = d(v)$ at each vertex $v$.

(R1) Each 2-vertex in a 2-thread gets a charge equal to $\frac{1}{2}$ from its $3^+$-vertex neighbor.

(R2) Each 2-vertex in a 1-thread gets a charge equal to $\frac{1}{4}$ from each of its neighbor.

(R3) Each $(2,0,1)$-vertex gets a charge equal to $\frac{1}{4}$ from its **slave**.

$\omega^*$ to denote the charge at each vertex $v$ after we apply the discharging rules. Note that the discharging rules do not change the sum of the charges. To complete the proof, we show that $\omega^*(v) \geq \frac{5}{2}$ for all $v \in V(G)$.

This leads to a contradiction.

**Contradiction**

$$\frac{5}{2} \leq \frac{\sum_{v \in V(G)} \omega^*(v)}{|V(G)|} = \frac{\sum_{v \in V(G)} \omega(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|} \leq Mad(G) < \frac{5}{2}.$$

# Conclusion

**Conjecture**

*Let $G$ be a graph with $g(G) \geq 2p+1$.*
*If $Mad(G) < 2 + \frac{1}{p}$, then $\chi_f(G) \leq 2 + \frac{1}{p}$*

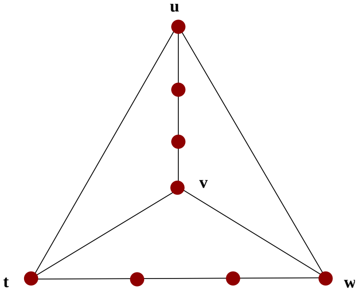
If this conjecture is true, then it is best possible.
Let be $G_p$ the complete graph $K_4$ with vertex set $\{t, u, v, w\}$. We replace the edge $uv$ by a path of length $2p-1$ and the edge $tw$ by a path of length $2p-1$.

# Conclusion

## Klostermeyer and Zhang ('00)

The graph $G_p$ cannot be $(2p+1, p)$-colored.

$odg(G_p) = 2p + 1$ and $Mad(G_p) = 2 + \frac{1}{p}$.




$G_2$