The uniform locally finite tilings of the plane

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Abstract

We characterize the uniform locally finite tilings of the plane by finite combinatorial objects called labeling schemes. This representation is based on the analysis of the group of symmetries of a tiling which is realized independently of its geometry. As a consequence, we are able to give an exhaustive enumeration of these tilings. This description encompasses the regular polyhedra and the Archimedean tilings, but most of these are tilings of the hyperbolic plane.

Key words: tiling of the plane, isogonal tiling, fuchsian group, group of symmetries, locally finite, labeling scheme

Introduction

A *tiling of the plane* (or simply a *tiling*) is a countable family of connected closed subsets of the plane $\mathcal{T} = \{T_1, \ldots, T_n, \ldots\}$ called *tiles*, such that:

(i) the union of the sets T_i is the whole plane;
(ii) the interiors of two distinct sets T_i and T_j are disjoint.

A *vertex* of a tiling is a point where at least three different tiles meet. A *symmetry* of a tiling is an isometry of the plane that leaves the tiling globally invariant. Grünbaum and Shephard [4] define an *isogonal* tiling as a tiling whose group of symmetries S is transitive on the set of vertices: for every pair of vertices, there exists a symmetry in S mapping the first onto the second. A *uniform* tiling is an isogonal tiling whose tiles are regular polygons. A *regular* tiling is a uniform tiling whose tiles are isometric copies of a given regular polygon. In the Euclidean plane, up to isometry and scaling, there

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exists exactly 11 uniform tilings (cf. [4] p. 63), 3 of which are regular. These tilings are called the Archimedean tilings.

Consider the problem of enumerating all the isogonal tilings of the plane. The nature of these tilings is strongly related to the geometry of the plane itself, when compared to other Riemannian surfaces, but also depends on the shape of the tiles, which could be unbounded, not simply connected or have a fractal boundary, or even the tiling themselves – depending on the arrangement of the tiles, there may exist accumulation points where an infinite number of tiles meet. First let us heed to the uniform case where every tile is a regular polygon, and catalog the 3 usual geometries :

(1) In the spherical case, the uniform tilings correspond exactly to the convex uniform polyhedra, containing the 5 Platonic solids, the 13 Archimedean solids and 2 infinite families of prisms and antiprisms [3].
(2) In the Euclidean case, the uniform tilings with finite faces are the Archimedean tilings [4]. Besides, if we consider half-planes to be infinite regular polygons, there exists 3 tilings with infinite faces: one corresponding to the limit of the prisms, one to the limit of the antiprisms, and one tiling made of two infinite faces.
(3) In the hyperbolic case, different techniques allow to construct various families of uniform tilings. For instance, the regular tilings of the hyperbolic plane are well known and can be represented using Schläfli symbols [3]. Moreover, the regular tilings of the plane with triangles can be represented by triangular symmetric tessellations [7,10]. This technique consists in drawing a triangle whose internal angles are submultiples of π, and considering the images of this triangle under the reflections induced by its edges. This approach is not exhaustive, since the groups of symmetries of these tilings are all necessarily generated by reflections. Another general method for enumerating uniform tilings of the plane is due to Huson [5,6]. It consists in representing tilings along with their groups of symmetries, by structures named Delaney-Dress symbols. These symbols allow to represent vertex-transitive tilings of the plane, yet is undermined by the fact that the group of symmetries of the tiling is necessarily a group of isometries of the plane.

Our intent is to provide a more general notion of symmetry to address particular families of isogonal tilings, in particular those containing unbounded tiles. In this article, we show that under mild restrictions on the topology of tilings, every isogonal tiling of the plane is combinatorially isomorphic to a uniform tiling. In addition, we give a characterization of the uniform tilings of the plane that is independent of the geometry, and that allows the construction of tilings with an arbitrary discrete group of symmetries of the plane. Our representation is based on an initial work of T. Chaboud [1], who proposed to represent the normal Cayley graphs by structures called labeling schemes.

These structures have been extended to transitive *topologically locally finite* planar (TLF-planar in short) graphs in [9,8].

First, we define the topological properties of the family of tilings that we study, namely the *topologically locally finite* tilings of the plane, and their possible symmetries. Next, we extend the concept of labeling scheme to allow the representation of the isogonal locally finite tilings of the plane. We show that such tilings can be embedded in a geometry of the plane such that all tiles are regular polygons (even the infinite faces). As a corollary, this representation allows us to enumerate all the uniform locally finite tilings having a given internal degree. Moreover, we can decide whether or not there exists a tiling possessing a particular local geometric property (for example the arrangement of the tiles around a given vertex).

1 Locally finite tilings

1.1 Tilings of the plane

Definition 1 (Plane): By the word of *plane*, we mean a simply connected Riemannian surface $\mathcal{P}$ without boundaries. $\diamond$

The definition ensures that a plane is locally isomorphic to the usual Euclidean plane. This is sufficiently general for our purposes, and we shall see that it is always possible to embed the tilings in one of the three usual geometries: the sphere, the Euclidean plane or the hyperbolic plane. An *isometry* of the plane $\mathcal{P}$ is a bijective map $\mathcal{P} \longrightarrow \mathcal{P}$ which preserves the distances. The three geometries cited above share analogies regarding their groups of isometries: all of them contain *reflections* (indirect isometries of order 2) and *rotations* (direct isometries of finite order) that are locally equivalent (meaning that their action on the tangent plane is the same [10]). It is therefore possible to speak of these isometries (a rotation around a given point or a reflection defined by a given geodesic), independently of the geometry of the plane.

Definition 2 (Tiling): A *tiling of the plane* (or simply a *tiling*) is a countable family of connected closed subsets of the plane $\mathcal{T} = \{T_1, \ldots, T_n, \ldots\}$ called *tiles*, such that:

(i) The union of the sets T_i is the whole plane.
(ii) The interiors of two distinct sets T_i and T_j are disjoint. $\diamond$

Since $\mathcal{P}$ is locally compact, we can consider the one-point compactification of the surface $\mathcal{P}$, denoted by $\bar{\mathcal{P}}$, where we add the point at infinity to the surface.

We assume the following restrictions on the tilings:

(*iii*) Each tile must be a closed topological disk in $\bar{\mathcal{P}}$. The boundary of a tile is a simple continuous curve in $\bar{\mathcal{P}}$.
(*iv*) The intersection of two different tiles is either empty or a finite set of isolated points and bounded $\mathcal{C}^1$ arcs [4].

The bounded arcs are called the *edges* of the tilings, the intersection points and the endpoints of the arcs are called the *vertices* of the tiling. The *border* of a tile is its topological boundary. It can be divided into a sequence of edges separated by vertices of the tiling. Every edge connects two vertices, called its *extremities*, and *separates* two tiles. The number of edges incident to a given vertex v is called the *degree* or the *valence* of v, and is at least 3. A *face* of the tiling is the topological interior of a tile of $\mathcal{T}$.

Definition 3 (Locally finite tiling): A tiling $\mathcal{T}$ is said to be *locally finite* if every compact subset of the plane meets only finitely many tiles of the tiling. As a consequence of (*iv*), every compact subset of the plane meets finitely many edges and vertices of $\mathcal{T}$, and the degree of every vertex is finite. $\mathcal{P}$ can therefore be partitioned into vertices, edges and *faces* (corresponding to the topological interior of the tiles) of $\mathcal{T}$. We shall write $\mathcal{T} = (V, E, F)$ where V is the set of vertices, E the set of edges and F the set of faces of $\mathcal{T}$. The *planar graph* associated to the tiling $\mathcal{T}$ is defined by the pair (V, E). Two examples of locally finite tilings are shown in figure 1. $\diamond$

Consider $\mathcal{T}$ a locally finite tiling of the plane. Let us analyze the local arrangement of the vertices, edges and faces around a given vertex. Two faces are said to be *adjacent* if their borders share a common edge or vertex. The *size* of a face is the (possibly infinite) number of vertices belonging to its border. A face of finite size is said to be *finite*, otherwise it is *infinite*. Since $\mathcal{T}$ is locally finite and the edges are bounded, the infinite faces are exactly the unbounded faces. Consider the set of faces incident to a vertex v, and let $[k_1, \ldots, k_d]$ be a vector describing the sizes of these faces, ordered in cyclic order around v. Such a vector is called a *type vector* or a *vertex type* for v. This vector is defined up to rotation and reflection. For the Euclidean regular tiling by squares, the type vector of a vertex is $[4; 4; 4; 4]$, whereas for the regular dodecahedron, it is $[5; 5; 5]$. Other examples are given in figure 1.

A *polygon* is a simply connected closed subset of the plane such that its border is a (possibly infinite) sequence of geodesics that are also called *edges*. A *regular polygon* is a convex polygon such that all its edges have same length and the internal angle between two consecutive edges is always the same. Infinite regular polygons in the Euclidean plane correspond to half-planes. In the hyperbolic plane, they are related to the orbits of limit rotations and translations [10]. All the faces of the tilings in figure 1 are regular polygons.

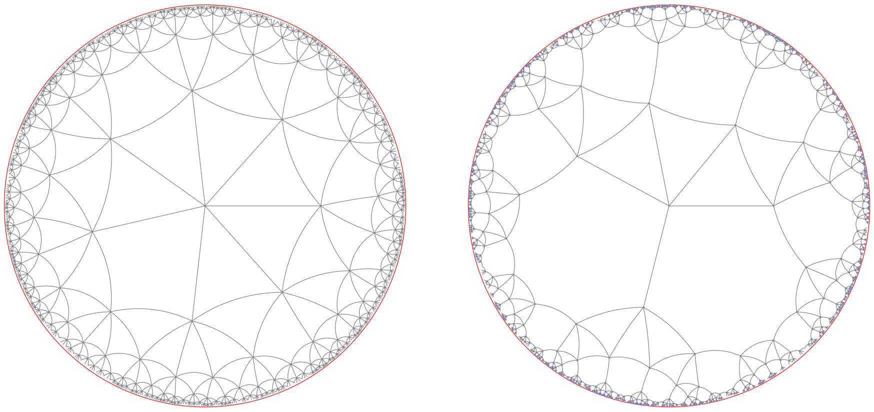


Fig. 1. Two locally finite tilings of the plane drawn in the hyperbolic plane (Poincaré model), with respective type vectors [3; 3; 3; 3; 3; 3; 4] and [3; 3; 3; 8; 8].

1.2 Automorphisms and symmetries

From now on, $\mathcal{T} = (V, E, F)$ will always be a locally finite tiling, and $G = (V, E)$ its associated planar graph.

Definition 4 (Automorphism): An *automorphism* of $\mathcal{T}$ is a bijective map φ defined on $V \cup E \cup \{f \in F, f \text{ is finite}\}$ such that:

(i) φ preserves the structure of the tiling: it maps a vertex onto a vertex, an edge onto an edge, and a finite face onto a finite face.
(ii) φ preserves the boundaries: if $e \in E$, the extremities of $\varphi(e)$ are the images of the extremities of e. If $f \in F$ and f is finite, the image of the border of f is the border of $\varphi(f)$. ◇

Note that automorphisms preserve both the graph associated to the tiling, and the adjacency relations of the finite faces. However, the image of an infinite face by an automorphism is not defined. More precisely, the image of the border of an infinite face may or may not be the border of another infinite face. A *strong automorphism* of $\mathcal{T}$ is an automorphism such that the image of the border of an infinite face is the border of another infinite face. We can extend the action of a strong automorphism onto the infinite faces of the tiling. A *weak automorphism* is an automorphism that is not strong. As an example, every isometry leaving the tiling invariant induces a strong automorphism. Tilings possessing weak automorphisms appear in figure 2 (p. 7) and figure 8 (p. 18). The set of strong automorphisms of $\mathcal{T}$ (resp. automorphisms) has a structure of a group under the operation of composition.

Definition 5 (Isogonal tiling): A tiling whose group of symmetries (resp. automorphisms) is *transitive* on its set of vertices is said to be *geometrically isogonal* (resp. *combinatorially isogonal*). Obviously, every geometrically isogonal tiling is also a combinatorially isogonal tiling. In the remaining of the paper, we shall only consider the combinatorially isogonal tilings and refer to them as the *isogonal tilings*. An isogonal tiling is said to be *strong* if it possesses only strong automorphisms, and *weak* otherwise. A *uniform* tiling is an isogonal tiling whose tiles are regular polygons. The *degree* of an isogonal tiling is defined as the degree of one of its vertices. ◇

The graph G associated to $\mathcal{T}$ is a topologically locally finite planar graph (in short TLF-planar) [8], and every automorphism of the tiling induces an automorphism of the graph. Therefore if $\mathcal{T}$ is isogonal, G is vertex-transitive. Conversely, the TLF-planar vertex-transitive graphs satisfy:

Theorem 1 (Preservation of the finite faces [9])
If G is a TLF-planar graph and $\mathcal{T}$ is the tiling induced by the embedding of G, the automorphisms of G preserve the borders of the finite faces of $\mathcal{T}$.
As a corollary, the automorphisms of G extend to automorphisms of $\mathcal{T}$.

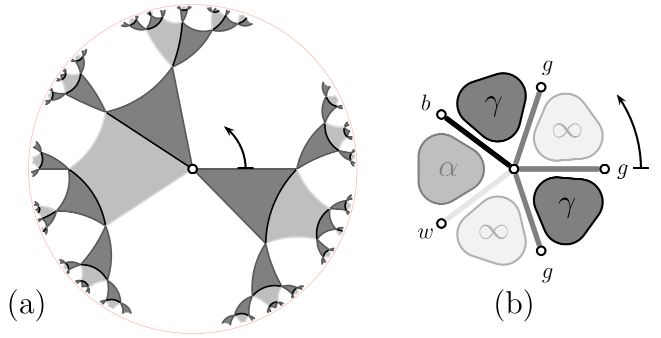
A *coloring* of $\mathcal{T}$ is an application $\delta : V \cup E \cup F \rightarrow \mathcal{C}$ where $\mathcal{C}$ is a finite set of *colors*. For the sake of simplicity, we require that $\delta(V)$, $\delta(E)$ and $\delta(F)$ are mutually disjoint. In the following, colors of faces appear as Greek letters $\{\alpha, \beta, \ldots\}$ and colors of edges as italic roman letters $\{b, g, w \ldots\}$ (these three letters corresponding respectively to black, gray and white).

For the remaining of the article, we let Γ be a transitive group of automorphisms of $\mathcal{T}$, and we define the following coloring of the tiling: two elements $(c_1, c_2) \in (V \cup E \cup \{f \in F, f \text{ is finite}\})^2$ have the same color if and only if there exists an automorphism in Γ mapping c_1 onto c_2. Since $\mathcal{T}$ is isogonal, all vertices have the same color, and the set of colors of edges and finite faces is bounded by the degree d of the tiling. Since the image of an infinite face by an automorphism is not properly defined, we assign as a convention a special color to all infinite faces. By definition, every automorphism preserves the colors. In the following, we will consider the tiling $\mathcal{T}$ colored by this particular coloring. In section 2, we show how this coloring characterizes the group Γ.

1.3 Edge and face vectors of $\mathcal{T}$

Let us consider the local action of the automorphisms on the tiling, and more specifically how an automorphism modifies the adjacencies between a vertex and its neighborhood in the tiling.

Definition 6 (Star): Let v be a vertex in V. The *star* centered on v is the set $\mathcal{R}[v] = \{v\} \cup \{x \in E \cup F, x \text{ is incident to } v\}$. The embedding of the tiling induces a cyclic ordering of the edges (resp. faces) around v. The ordered set of color of edges (resp. faces) of $\mathcal{R}[v]$ is called the *edge vector* (resp. *face vector*) around v (an example appears in figure 2). ◇



Edge and face vectors

ξ	:	$[g; g; b; w; g]$
ϕ	:	$[\infty; \gamma; \alpha; \infty; \gamma]$

(c)

Fig. 2. A locally finite tiling (a), the star (b) centered on the central vertex of the tiling and the associated edge and face vectors (c). Faces labeled by α are squares, those labeled by γ are triangles and the remaining ones are infinite.

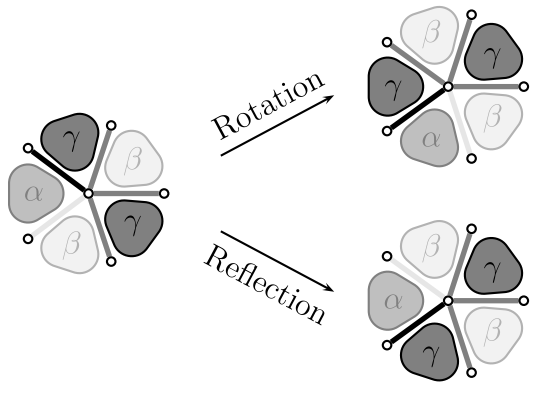
Consider a vertex $v \in V$ and the star $\mathcal{R}[v]$. A *block of finite faces* around v is a finite sequence $\{e_0, f_0, e_1, f_1, \dots f_{n-1}, e_n\}$ where the e_i are edges in $\mathcal{R}[v]$, the f_j are finite faces in $\mathcal{R}[v]$, and they appear consecutively around v. Such a block is said to be *maximal* if it is not a strict subset of a larger block. Now consider an automorphism φ mapping v onto v'. Since φ preserves the adjacencies, we have:

(*i*) If an edge e separates two *finite* faces f and f', then $\varphi(e)$ separates $\varphi(f)$ and $\varphi(f')$ and *vice versa*.

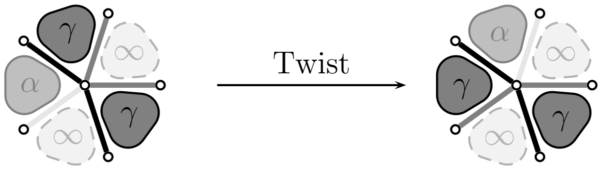
(*ii*) If a *finite* face f separates two edges e and e' around v, their images $\varphi(e)$ and $\varphi(e')$ are separated by $\varphi(f)$ and *vice versa*.

Therefore, the group of automorphisms of $\mathcal{T}$ acts on the set of stars of $\mathcal{T}$ and this action maps the maximal blocks of finite faces attached to v onto maximal blocks of finite faces attached to v'. We can categorize the action of the automorphisms into two subsets, depending on whether the automorphism is weak or strong:

- *Strong automorphisms*: these automorphisms preserve all the adjacencies of the edges and faces around a vertex, hence they correspond to rotations and/or reflections of the star centered on v;



- *Weak automorphisms*: conversely, these automorphisms do not necessarily preserve all the adjacencies around a vertex. However, they must preserve the maximal blocks of finite faces attached to v. A transformation that modifies the arrangement of the maximal blocks around v without being a simple composition of rotations and reflections is called a *twist*.



  For example, supposing here that the faces α and γ are finite, the transformation φ reverses the block of finite faces containing $\{\alpha; \gamma\}$ while leaving the block containing $\{\gamma\}$ unchanged. Notice that the twists can only happen when there exists at least two infinite faces incident to the vertex v.

As a consequence, we can state the following lemma:

Lemma 2 (Local action of the automorphisms (1))
Let $\mathcal{T}$ be an isogonal locally finite tiling. For every pair of stars $(\mathcal{R}[v], \mathcal{R}[v'])$ in $\mathcal{T}$, there exists a finite sequence of rotations, reflections and twists that transforms $\mathcal{R}[v]$ into $\mathcal{R}[v']$.

Concretely, depending on $\mathcal{T}$, there exists a finite set of transformations that acts transitively on the stars of $\mathcal{T}$. Let's select a vertex $v \in V$ and define $\mathcal{R}_{\mathcal{T}} = \mathcal{R}[v]$ as the star of the tiling. The choice of the vertex v is not important, since these vectors are defined up to transformation by a rotation, reflection or a twist. Consequently, we define the *edge vector* (noted ξ) and the *face vector* (noted ϕ) of $\mathcal{T}$ respectively as the edge vector and face vector of $\mathcal{R}_{\mathcal{T}}$. To determine entirely $\mathcal{R}_{\mathcal{T}}$, we number each edge and each face from 1 to d, and the star $\mathcal{R}_{\mathcal{T}}$ is represented by the pair of vectors (ξ, ϕ), with the convention that the edge ξ_i separates the faces ϕ_i and $\phi_{(i-1 \bmod d)}$.

Our goal is to give a combinatorial representation of an isogonal tiling $\mathcal{T}$ and the coloring induced by a transitive group of automorphisms Γ. This representation contains an edge vector, a face vector and a type vector that specifies the sizes of the faces. Yet these 3 vectors are not sufficient to completely determine the colored tiling. Figure 3 gives an example of tiling $\mathcal{T}$ with two transitive groups of automorphisms Γ and Γ'. The two colorings of Γ induce the same edge and face vector, but the colored tilings are not isomorphic. Indeed, we can compare the borders of the faces in each embedding (cf. section 2.2), and show that the border of the faces for Γ cannot be mapped onto the border of the faces for Γ'.

The edge, face and type vectors represents three geometrical invariants for the vertices of $\mathcal{T}$. But, according to the previous example, this information is not sufficient to characterize the tiling and its coloring. Therefore, it is natural to look for an additional information related to the edges of $\mathcal{T}$.

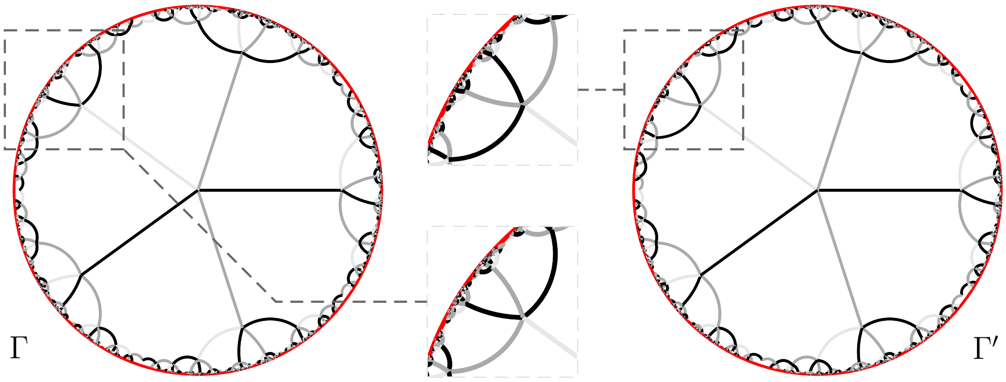


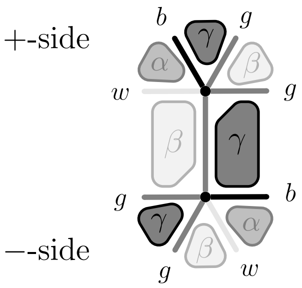
Fig. 3. Two representations of the hyperbolic regular tiling with type vector $[10; 10; 10; 10; 10]$ (with Schläfli symbol $\{10, 5\}$), with two transitive groups of automorphisms Γ and Γ'. Both groups act transitively on the faces of $\mathcal{T}$. Both colorings possess the same edge and face vectors, but the groups are not isomorphic. One difference between colorings is the fact that the extremities of the lighter edge are non-isomorphic (see detail). Should both colored tilings be isomorphic, an isomorphism would preserve the borders of the faces, but their borders are non-isomorphic sequences of colors: $(wgbgb)^2$ (for Γ) and $(wgbgbwbgbg)$ (for Γ').

1.4 Edge neighborhoods of $\mathcal{T}$

Definition 7 (Edge neighborhood): Let e be an edge in E. The *edge neighborhood* $\mathcal{E}[e]$ centered on e is defined as the pair of edge and face vectors attached to both extremities of e along with the relations of adjacence between edges and faces. It is represented by two pairs of edge and face vectors (ξ^+, ϕ^+) and (ξ^-, ϕ^-) such that:

- The i-th edge $\xi_i^\pm$ separates the faces $\phi_i^\pm$ and $\phi_{i-1}^\pm$.
- The first edge in the edge vectors is the common edge e: $\xi_1^+ = \xi_1^-$.
- The first and last face in each face vector correspond: $\phi_1^+ = \phi_d^-$ and $\phi_d^+ = \phi_1^-$. ◇

An example of edge neighborhood is given on figure 4.

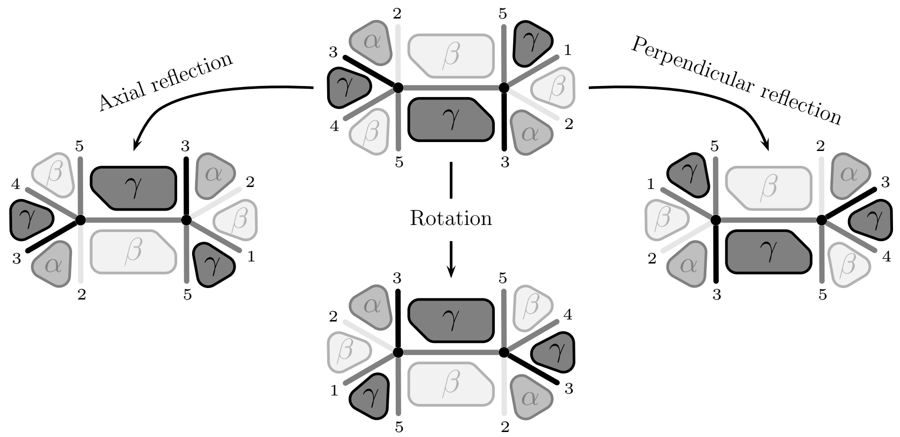


Vector representation		
ξ^+	:	$[g; g; g; b; w]$
ϕ^+	:	$[\gamma; \beta; \gamma; \alpha; \beta]$
ξ^-	:	$[g; g; g; w; b]$
ϕ^-	:	$[\beta; \gamma; \beta; \alpha; \gamma]$

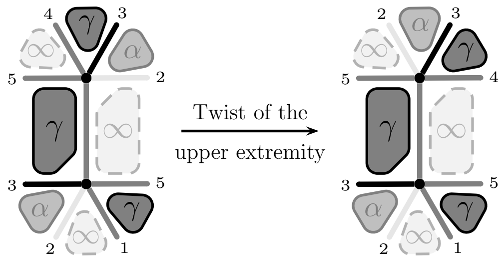
Fig. 4. The edge neighborhood associated to a gray edge of the tiling shown in figure 2, and its representation in terms of edge and face vectors.

Consider φ an automorphism, mapping the edge e onto the edge e'. As in Lemma 2, we consider the action of the group of automorphisms on the set of edge neighborhoods of $\mathcal{T}$. Since φ preserves the boundaries, it maps the edge neighborhood $\mathcal{E}[e]$ onto the edge neighborhood $\mathcal{E}[e']$. It induces a transformation for the stars centered at each extremity of e. Depending on whether the automorphism is strong or weak, φ belongs to one of the following subsets:

- *Strong automorphisms*: these correspond exactly to the reflections and rotations exchanging the extremities of e or not;



  The axial reflection corresponds to the reflection through the edge, the perpendicular reflection corresponds to the reflection through the perpendicular bisector of that edge.
- *Weak automorphisms*: Since an automorphism preserves the blocks of finite faces around the extremities of e, a weak automorphism corresponds to a twist of any or each extremity of e, possibly followed by a perpendicular reflection.



$\square$

Following this enumeration, we have:

Lemma 3 (Local action of the automorphisms (2))
Let $\mathcal{T}$ be an isogonal colored locally finite tiling, and $\mathcal{E}[e]$ be an edge neighborhood of $\mathcal{T}$. Let $\mathcal{E}[e']$ be an edge neighborhood of an edge of the same color. Then there exists a finite sequence of rotations, reflections and twists that transforms $\mathcal{E}[e]$ into $\mathcal{E}[e']$.

The action of the automorphisms of the tiling on the set of edge neighborhoods can therefore be described using rotations, reflections and twists. Consider the equivalence relation $\approx$ defined on the set of the edge neighborhoods by the orbits resulting from the action of Γ. By Lemma 3 and the definition of the colors, each class of equivalence corresponds exactly to one color of edge. Therefore, for each color c of edges in $\mathcal{T}$, it is possible to define a representative edge neighborhood $\mathcal{E}_{\mathcal{T}}^{c}$.

An edge neighborhood $\mathcal{E}[e]$ is said to be *compatible* with the edge and face vectors of $\mathcal{T}$ if both stars at the extremities of $\mathcal{E}[e]$ are equivalent to $\mathcal{R}_{\mathcal{T}}$ up to an automorphism. In an isogonal tiling $\mathcal{T}$, for every edge color c, the edge neighborhood $\mathcal{E}_{\mathcal{T}}^{c}$ must be compatible with the edge and face vectors of $\mathcal{T}$. Remember that we numbered the edges and faces of $\mathcal{R}_{\mathcal{T}}$ in the previous paragraph. For simplification and as a consequence from Lemma 3, we can select the representative $\mathcal{E}_{\mathcal{T}}^{c}$ such that the edges at each extremity are numbered in a cyclic order $\{1, 2, \ldots, d\}$ around each extremity (possibly by applying some twist transformations, see figure 5). Now, the edge neighborhoods of the tiling depict the way the stars of the tiling are "linked" together.

Definition 8 (Opposite, inverse): In the tiling $\mathcal{T}$ and for every i in $\{1; 2; \ldots d\}$, we define the *opposite* and the *inverse* of the i-th edge of $\mathcal{T}$ in the following manner: let c be the color of the i-th edge in $\mathcal{R}_{\mathcal{T}}$, and $\mathcal{E}_{\mathcal{T}}^{c}$ the associated edge neighborhood. Since $\mathcal{E}_{\mathcal{T}}^{c}$ is compatible with $\mathcal{R}_{\mathcal{T}}$, at least one extremity of $\mathcal{E}_{\mathcal{T}}^{c}$ is equivalent to $\mathcal{R}_{\mathcal{T}}$ such that the central edge of $\mathcal{E}_{\mathcal{T}}^{c}$ be the i-th edge of $\mathcal{R}_{\mathcal{T}}$. Let $\mathcal{R}'$ be the other extremity of the edge neighborhood. The *opposite* of i is defined as the index of the central edge of $\mathcal{E}_{\mathcal{T}}^{c}$ in $\mathcal{R}'$. The *inverse* of i is the element in $\{+; -\}$, "+" meaning that the maximal block of finite faces in $\mathcal{R}'$ containing the central edge is numbered in the same order of rotation as in $\mathcal{R}_{\mathcal{T}}$, and "−" otherwise. An example of the computation of these functions is given in figure 5. ◇

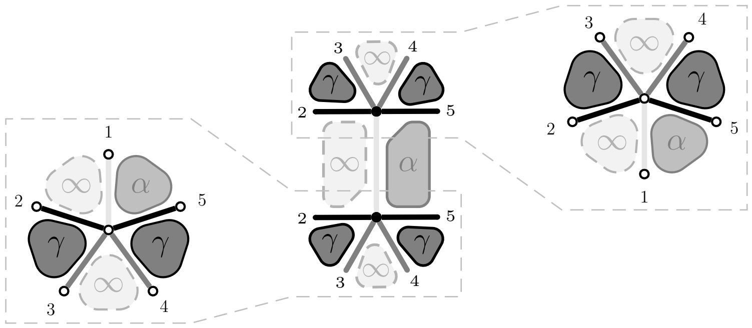


Fig. 5. Given the above star and edge neighborhood, the opposite of 1 is 1 and the inverse of 1 is "−". Indeed, for the maximal block of finite faces $\{\alpha; \gamma\}$ the direction of rotation on the left and on the right star differ.

Notice that the previous functions depend on the choices made when selecting the edge neighborhoods and the stars up to the corresponding relations of equivalence. Therefore they characterize only the set of representative edge neighborhoods. Following the definition, it is possible to define an "opposite" vector and an "inverse" vector for the tiling.

2 Labeling schemes

2.1 Definition and abstract representation

A labeling scheme is a combinatorial object used in [9,8] to give a description of transitive planar graphs. In this section, we define another kind of labeling scheme to represent isogonal locally finite tilings.

Definition 9 (Labeling scheme): A *labeling scheme* of degree d is a 3-tuple (ξ, ϕ, η) such that:

- ξ is an edge vector of length d, and ϕ is a face vector of length d, edges and faces being represented by their colors.
- $\eta = \{\mathcal{E}_{c_1}, \ldots, \mathcal{E}_{c_k}\}$ is a finite set of edge neighborhoods, such that for every color of edge $c \in \xi$, there exists exactly one edge neighborhood $\mathcal{E}_c \in \eta$, whose central edge is colored by c and which is compatible with (ξ, ϕ). $\diamond$

Starting from an isogonal locally finite tiling $\mathcal{T} = (V, E, F)$, sections 1.3 and 1.4 detailed the construction of a labeling scheme associated to $\mathcal{T}$ by selecting representative edge and face vectors, and a finite set of representative edge neighborhoods. The tiling $\mathcal{T}$ is said to be *colored* by this labeling scheme.

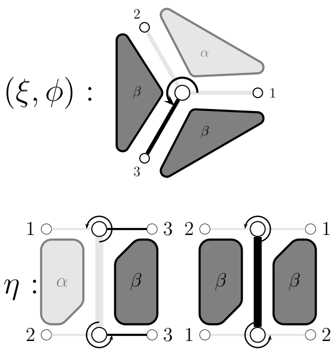
Since this representation is not practical to handle for algorithmic purposes, we define an equivalent representation called the *abstract representation* of the labeling scheme as a 4-tuple $(\xi, \phi, \mathrm{opp}, \mathrm{inv})$ of vectors of the same length:

- Two vectors corresponding to ξ and ϕ, with the convention that the edge ξ_i separates the faces ϕ_i and $\phi_{(i-1 \bmod d)}$.
- A vector opp, called the *opposite* vector of the scheme, assigning to every edge of ξ its opposite in ξ according to the set of edge neighborhoods η;
- A boolean vector inv, called the *inverse* vector of the scheme, indicating for each edge in ξ its inverse in ξ according to the set of edge neighborhoods η.

In these vectors, the colors are given by integers and the directions of rotation $\{+, -\}$ are given by booleans. An example of labeling scheme and its abstract representation appears in figure 6. The scheme and its abstract representation both represent the tiling $\mathcal{T}$, but the abstract representation can directly be used inside algorithms, such as for example the Listing 1.

2.2 Border automaton

The labeling scheme (ξ, ϕ, η) describes the local arrangement of edges and faces of $\mathcal{T}$ around vertices and edges, but it does not directly specifiy the sizes

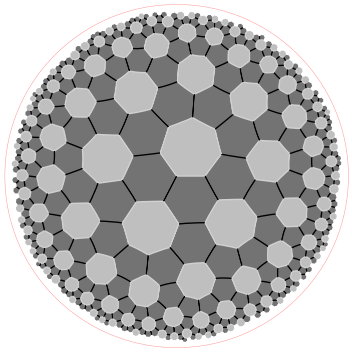


(a)

	1	2	3
Vector ξ :	1	1	2
Vector ϕ :	1	2	2
Vector opp :	2	1	3
Vector inv :	$+$	$+$	$+$

In the tiling, the color 1 is light and the color 2 is dark.

(b)



(c)

Fig. 6. A labeling scheme defined by (a) the vectors (ξ, ϕ) and the two neighborhoods η, (b) its abstract representation and (c) the tiling from which it is constructed.

of the faces themselves, or more precisely the type vector of $\mathcal{T}$. Thus, given a labeling scheme, there may exist several tilings $\mathcal{T}$ with different type vectors possessing the same labeling scheme. Starting from (ξ, ϕ, η), we construct a set of *valid type vectors*, meaning the set of type vectors of the tilings that are colored by this labeling scheme. According to [8], there exists a special kind of finite state automaton called the *border automaton* of the tiling that is built using techniques similar to those used for maps and hypermaps [2]. This automaton describes the borders of the faces of a tiling colored by (ξ, ϕ, η). Let us now detail its construction. Recall that d is the degree of the tiling.

Definition 10 (Handle): Consider a tiling $\mathcal{T}$ colored by (ξ, ϕ, η), and $\mathcal{R}_\mathcal{T}$ the star of the tiling. A *handle* is a pair (k, b), where $k \in [1; d]$ and $b \in \{+, -\}$. This handles is meant to describe a direction of rotation b for $\mathcal{R}_\mathcal{T}$, and the index of a face attached to the k-th edge in $\mathcal{R}_\mathcal{T}$: the k-th face if b is $+$, otherwise the $(k-1)$-th face). Two pairs (k, b) and (k', b') are equivalent if there exists a rotation, reflection or twist mapping the face (k, b) and the k-th edge onto the face (k', b') and the k'-th edge. ◇

In the tiling in figure 6, the handles $(2, +)$ and $(1, -)$ are equivalent.

Now let f be a face of the tiling, and suppose that we want to describe the border of f by computing the (possibly bi-infinite) word of colors of the edges along its border in the positive direction of rotation. The sequence of vertices in the border of f is noted $\{\ldots, v_{-1}, v_0, v_1, v_2, \ldots\}$. For each vertex v_i, it is possible to define a handle (k_i, b_i) that defines the position of the face f around v_i and the direction of rotation of the faces around v_i. The edge neighborhood corresponding to the edge between v_i and v_{i+1} defines a relation between (k_i, b_i) and (k_{i+1}, b_{i+1}).

Definition 11 (Border automaton): The *border automaton* $\mathcal{A}$ of the labeling scheme is a finite state automaton, whose vertices are the handles, and whose transitions are defined as follows: for every edge neighborhood $\mathcal{E}_c$ in η, and every face f incident to the central edge of $\mathcal{E}_c$, the are two handles (k_u, b_u) and (k_d, b_d) specifying the face f on each extremity of $\mathcal{E}^c$. If we follow the border of f in any direction (positive or negative), this defines a transition from one handle to the other. ◇

By definition, the border of a face corresponds to a sequence of transitions in $\mathcal{A}$. For every state (k, b) of the automaton, consider an edge neighborhood $\mathcal{E}_0 \in \eta$ whose central edge is the k-th edge in η and whose star at the lower extremity is labeled by $[\xi_k, \xi_{k+1} \ldots \xi_d, \xi_1 \ldots, \xi_{k-1}]$ (according to definition 7). A transition in $\mathcal{A}$ starting from the state (k, b) corresponds to following the border of one of the faces of $\mathcal{E}_0$. Hence every state has at least one outgoing edge, and if it has more than one, this implies that there are more than one possible continuations for the border of this face (resulting from a twist in $\mathcal{E}_0$), and that it corresponds to the handle of an infinite face.
Then it is possible to study the connected components in $\mathcal{A}$. The border of a finite face corresponds to a finite sequence of cycles of the automaton, and the border of an infinite face is a bi-infinite walk in the automaton.

Example 1 (Construction of a border automaton): Starting from the tiling $\mathcal{T}$ drawn in figure 2 on page 7, it is possible to build the labeling scheme and the border automaton appearing in figure 7. In this example, there are two disjoint cycles of length 2 and 3, and a connected component containing states with more than one outgoing edges.
A cycle of length 2 corresponds to a face of size $2n$ (for $n \geq 2$), and a cycle of length 3 corresponds to a face of size $3m$ (for $m \geq 1$). There is only one kind of bi-infinite walk, corresponding to an infinite face whose border is an infinite sequence of edges labeled by 2 (gray), followed by an edge labeled by 4 (light) and then an infinite sequence of edges labeled by 1 (gray).

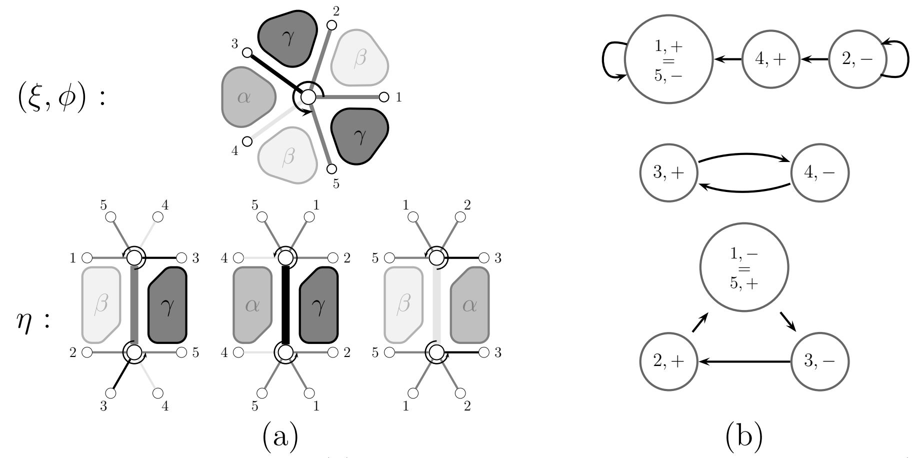


Fig. 7. The labeling scheme (a) associated to the tiling in figure 2 and (b) the border automaton that can be constructed from it. The valid type vectors for this automaton belong to the set $[\infty; 3n, 2m; \infty; 3n]$, where the faces labeled γ are of size $3n$, the faces labeled α are of size $2m$ and the faces labeled β are infinite.

Then for every $i \in [1;d]$, it is possible to compute whether the border of the face starting from the handle $(i,+)$ remains inside a finite cycle of length k_i or belongs to an bi-infinite walk. In the first case, the size of the face is a multiple of k_i (possibly infinite), and in the second case, the face is necessarily infinite. In this example, the *valid type vectors* for the tiling are therefore the set $[\infty; 3n, 2m; \infty; 3n]$, for values of $m \geq 2$ and $n \geq 1$. ∎

2.3 Construction of uniform tilings

To characterize the tiling up to the transformations described in the previous paragraphs, one needs a labeling scheme and a valid type vector which has been computed from the border automaton. We will now show that this representation is sufficient to construct a uniform tiling possessing the same edge, face and type vectors.

Theorem 4 (Existence) *Given a labeling scheme (ξ, ϕ, η), and a valid type vector $[k_1; \dots; k_d]$, there exists a uniform locally finite tiling of the plane possessing this scheme and type vector. Moreover, all faces of the tiling are regular polygons.*

Proof: Let $\mathcal{A}$ be the border automaton associated with the labeling scheme. First let us consider the choice of the geometry of the plane in which we are going to construct the tiling. Compute $\Delta = \sum_{i=1}^{d} \frac{k_i-2}{k_i}\pi$. Depending on the value of Δ, we select an appropriate geometry: if $\Delta < 2\pi$, we consider the sphere, if $\Delta = 2\pi$, we consider the Euclidean plane and when $\Delta > 2\pi$, we consider the hyperbolic plane.

It is then possible to compute regular polygons P_i of size k_i in the selected geometry. The internal angle $\theta_i(l)$ of a regular polygon of size k_i, whose edge has length l, is given by the following equation depending on the choice of the geometry:

$$\theta_i(l) = \begin{cases} \text{Spherical}: & 2\arcsin\left(\frac{\cos(\pi/k_i)}{\cos(l/2)}\right) \\ \text{Euclidean}: & \frac{(k_i-2)}{k_i}\pi \\ \text{Hyperbolic}: & 2\arcsin\left(\frac{\cos(\pi/k_i)}{\cosh(l/2)}\right) \end{cases}$$

The equation $\sum_{i=1}^{d} \theta_i(l) = 2\pi$ has a solution for l that allows to build the regular polygons P_i of size k_i such that around any point p of the plane, the polygon P_i and P_{i+1} share a single edge incident to p. This allows to build the geometrical embedding of a star $\mathcal{R}[p]$ centered on the point p, without gaps or overlapping of the faces.

Let us color $\mathcal{R}[p]$ adequately. We label the edges and the faces incident to p according to the vectors (ξ, ϕ). Moreover, since the type vector is valid, it is possible to label the borders of the polygons so that they correspond to the borders computed by the border automaton. And in the graph induced by this arrangement of polygons, the stars of all vertices except the central vertex are subsets of (ξ, ϕ), meaning that they are a set of adjacent edges and faces of (ξ, ϕ). The same property is valid for the edge neighborhoods.

We now begin the construction of the tiling by induction. Let ϵ be a point of the plane, and $\mathcal{R}[\epsilon]$ the star built around ϵ according to the previous paragraph. Let G_1 be the graph containing all the vertices and edges of the polygons in $\mathcal{R}[\epsilon]$. By induction, we consider the graph G_n, for which all the stars are subsets of (ξ, ϕ), and we build a graph $G_{n+1} \supset G_n$ whose faces in the embedding are all regular polygons. Consider a vertex v in G_n and $\mathcal{R}[v]$ the star centered on v. If there is less that d polygons in this star, it is possible to build the missing ones around v without gaps nor overlapping. We label these new polygons so that $\mathcal{R}[v]$ be equivalent to (ξ, ϕ). The border automaton indicates how these polygons must be labeled and arranged around v, in such a way that the edge and face vector of v correspond to (ξ, ϕ). We add these new polygons to G_{n+1} and iterate the construction.

Since eventually every point of the plane falls into some G_n, this construction defines a limit tiling $\mathcal{T}$ possessing the adequate type vector and labeling scheme. Then considering the tiling $\mathcal{T}$, we can restart the same construction from any vertex v of $\mathcal{T}$, meaning that we construct a new sequence $G_n[v]$ by reusing the edges and faces of $\mathcal{T}$. Since for every n, G_n and $G_n[v]$ are isomorphic, we construct an automorphism of $\mathcal{T}$ mapping ϵ onto v. Therefore the tiling $\mathcal{T}$ is uniform. □

The construction described in the previous proof is effective and we give a description of the algorithm constructing finite balls of the tiling. To every vertex v of the tiling, we attach a boolean $v.dir$ indicating the direction of rotation of $\mathcal{R}_{\mathcal{T}}$ around v, and an angle $v.first$ indicating the direction of the edge numbered 1 in $\mathcal{R}_{\mathcal{T}}$ and attached to v. Given a vertex v, the labeling algorithm (detailed in Listing 1) constructs and labels all edges incident to a given vertex v, and prepares the labeling of the vertices at the extremities of these edges. Then, to construct a ball of radius R of the tiling, it is sufficient to construct recursively the vertices up to distance R using the labeling algorithm.

2.4 Distinction between strong and weak automorphisms

In this paragraph, we highlight the differences between strong and weak automorphisms at the level of the tiling itself. First notice that the strong automorphisms are quite simple to describe since they necessarily correspond to isometries:

Data : $(\xi, \phi, \text{opp}, \text{inv})$ the abstract representation of a labeling scheme
$[k_1, \ldots, k_d]$ a valid type vector
v the central point for the star

Compute the length l and the list of angles $[\theta_1(l), \ldots, \theta_d(l)]$
For $i \in [1; d]$ **Do** // *For all edges around* v
  $\beta \leftarrow \sum_{j=1}^{i-1} \theta_j(l)$
  Build an edge e_i of length l starting from v, with angle β to $v.first$ in the direction indicated by $v.dir$.
  Let v_i be the other extremity of e_i
  If $b = +$ **Then** $v_i.dir \leftarrow \text{inv}(i)$ **Else** $v_i.dir \leftarrow not(\text{inv}(i))$
  // *From the point of view of* v_i, e_i *is indexed by* opposite(i)
  If $v_i.dir = +$ **Then**
    $v_i.first \leftarrow v.first + \pi - \sum_{j=1}^{\text{opp}(i)-1} \theta_j(l)$
  Else
    $v_i.first \leftarrow v.first + \pi + \sum_{j=1}^{\text{opp}(i)-1} \theta_j(l)$
  Label the edge e_i with the color $\xi(i)$
  Label the face between e_i and e_{i+1} with the color $\phi(i)$
EndFor

Listing 1. Constructing the star around a vertex

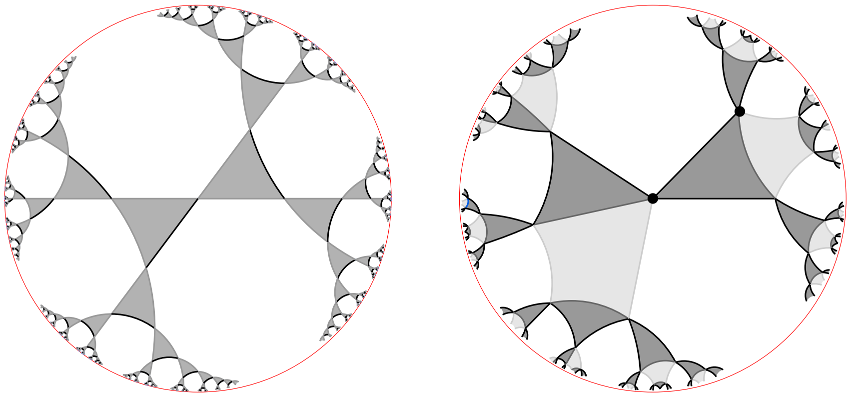
Lemma 5 *In a tiling of the plane with regular polygons, the strong automorphisms are isometries.*

Proof: Remember that φ preserves the faces and their borders. Since all faces are regular polygons, the restriction of φ to a face maps a regular polygon onto a regular polygon of the same size, and is therefore an isometry. Since φ also preserves the adjacencies of the faces, the restriction of φ to the stars of the tiling, φ is also an isometry. Consider now a line segment $[a; b]$ drawn in the plane, and partition this segment into $\{a_0 = a; a_1; \ldots; a_n = b\}$, such that every segment $[a_i; a_{i+1}]$ is entirely included into a face of the tiling, and that no face contains two different segments. For every $i \in [1; n-1]$, the segment $[a_{i-1}; a_{i+1}]$ is included into two adjacent faces belonging necessarily to a star of the tiling. Therefore, $\varphi([a_{i-1}; a_{i+1}])$ is also a line segment of the same length. By summation, $\varphi([a; b])$ is a line segment of the same length than $[a; b]$, and φ is an isometry. □

Consequently, if the group of automorphisms contains only strong automorphisms, it is isomorphic to a discrete group of isometries of the plane. Now the weak automorphisms preserve at least the finite faces, upon which they act like isometries. Consider the following examples.

Example 2 (Tilings possessing weak automorphisms): The tilings of the hyperbolic plane in figure 8 have type vector $[3; \infty; 3; \infty]$ and $[3; \infty; 3; 4; \infty]$.

The tiling (a) possesses weak automorphisms, and the coloring in the figure defines a transitive group of weak automorphisms. Nevertheless, this tiling also possesses a transitive group containing only strong automorphisms. On the contrary, the tiling (b), which is the example given in the figures 2 and 7, does not possess any transitive group of strong automorphisms. ▬



(a) $[3; \infty; 3; \infty]$ (b) $[3; \infty; 3; 4; \infty]$

Fig. 8. Two colored tilings of the hyperbolic plane possessing weak automorphisms. For the tiling (b), the automorphisms exchanging the two pointed vertices are necessarily weak.

In fact, it is possible to point out a family of uniform tilings whose transitive groups of automorphisms necessarily contain weak automorphisms, and this by simply looking at the border of the infinite faces:

Lemma 6 *If a uniform tiling $\mathcal{T}$ possesses infinite faces whose border is not periodic, then its group of automorphisms contains a weak automorphism.*

Proof: Suppose that there exists a transitive group of strong automorphisms Γ for the tiling $\mathcal{T}$. By definition, the group Γ acts on the infinite faces of $\mathcal{T}$. Let f_0 be a face whose border is not periodic. An automorphism mapping the border of f_0 onto itself is either trivial or a reflection, hence the stabilizer of the border of f contains at most 2 elements.

Let $\mathcal{O}$ be the orbit of f_0 under Γ. Consider the action of Γ on the pairs (f', v') where $f' \in \mathcal{O}$ and v' is a vertex of the tiling in the border of f'. Intuitively, the couple (f', v') represents a handle for designating the face f' attached to v'.

Since Γ is vertex-transitive, the orbit of the pair (f', v') contains a handle (f'_v, v) for every vertex v of the tiling. If d is the degree of $\mathcal{T}$, there cannot exist more than d handles per vertex, hence there exists at most d orbits in this action.

Now, consider the set $\{(f_0, v'), v'$ belongs to the border of $f_0\}$. There exists an orbit occurring infinitely often inside this set. And every element of this orbit preserves the border of f_0. This is impossible since the stabilizer of the border of f contains at most 2 elements. □

For a given degree d, consider the problem of finding all the possible uniform tilings of the plane with internal degree d:

Theorem 7 (Enumeration – Schemes)
Given $d \geq 2$, it is possible to enumerate all labeling schemes of degree d.

Proof: Consider the set $\mathcal{S}$ of labeling schemes of degree d. The possible edge and face vectors of the schemes in $\mathcal{S}$ contain at most d distinct colors. Moreover, given an edge and a face vector, a full set of edge neighborhoods for the scheme can be defined with the help of a next and an inverse vector, corresponding to at most $(2d)^d$ different sets. The number of schemes of degree d is therefore bounded, and these schemes can be enumerated along with their border automata. $\square$

Corollary 8 (Enumeration – Tilings)
Given $d \geq 2$, there exists an algorithm that enumerates all the uniform locally finite tilings of the plane having internal degree d.
For every of these tilings $\mathcal{T}$, the algorithm enumerates the finite set of colorings that correspond to a transitive group of automorphisms.
Moreover, it is decidable whether any of these groups contains weak automorphisms or not.

Notice that non-isomorphic colorings of the tiling do not necessarily correspond to non-isomorphic groups of automorphisms. The two following presentations correspond to the crystallographic group $p2$, and their Cayley graph is the Euclidean grid:

$$\left\{a_1, \ldots, a_4 \middle| a_1^2, a_2^2, a_3^2, a_4^2, a_1a_2a_3a_4\right\} \text{ and } \left\{a_1, \ldots, a_3 \middle| a_1^2, a_3^2, (a_2a_3)^2, (a_2^{-1}a_1)^2\right\}$$

Nevertheless, the first leads to a coloring of the grid with 4 colors of edges, while the second possesses only 3 colors of edges.

Example 3 (General enumeration): In figure 10, we enumerate all the possible uniform locally finite tilings of the plane whose internal degree is 5. These correspond to 52 labeling schemes, among which 4 correspond to weak tilings of the plane. For each of these tilings, we also give the set of valid type vectors. As an example, for the uniform tiling $[5;5;5;5;5]$ of Schläfli symbol $\{5,5\}$, there exists 9 non-isomorphic colorings corresponding to a transitive group of automorphisms. All of them are face-transitive. A partial enumeration of the 605 labeling schemes of degree 7 appears in figure 11, and other enumerations can be found in [8]. ▬

For each labeling scheme, the set of valid type vectors defines an infinite family of tilings that can be colored by the labeling scheme. Every one of these families contains tilings with infinite faces (and in particular the tiling of the plane of degree d with only infinite faces). In these enumerations, the tilings possessing weak automorphisms form a small minority, but they appear for every degree $d \geq 4$ (the only possibility for degree 4 appearing in figure 8).

Example 4 (Enumeration with geometric constraints): The computation of Δ tells us which geometry is adequate for a given tiling. If $\Delta < 2\pi$ the tilings are embedded in the sphere, and if $\Delta = 2\pi$, in the Euclidean plane. Consider the problem of finding the tilings whose value of Δ is strictly superior to 2π, but is minimal among all possible values of Δ for hyperbolic uniform locally finite tilings.

For a uniform tilings of degree ≥ 7, $\Delta \geq 7/3\pi$ (a value that is obtained only with the regular tiling $\{7, 3\}$). Therefore, to find this minimal value, we need to consider only the tilings of degree ≤ 6. This amounts to a finite number of labeling schemes. If we restrict our attention to tilings with finite faces, an exhaustive analysis (similar to the enumeration of the discrete groups of isometries of the sphere) provides the following type vectors:

- $[4; 6; 14]$ ($\delta = \Delta - 2\pi = \pi/42$),
- $[3; 14; 14]$, $[7; 6; 6]$, $[3; 4; 4; 7]$, and $[3; 3; 3; 3; 7]$ ($\delta = \pi/21$).

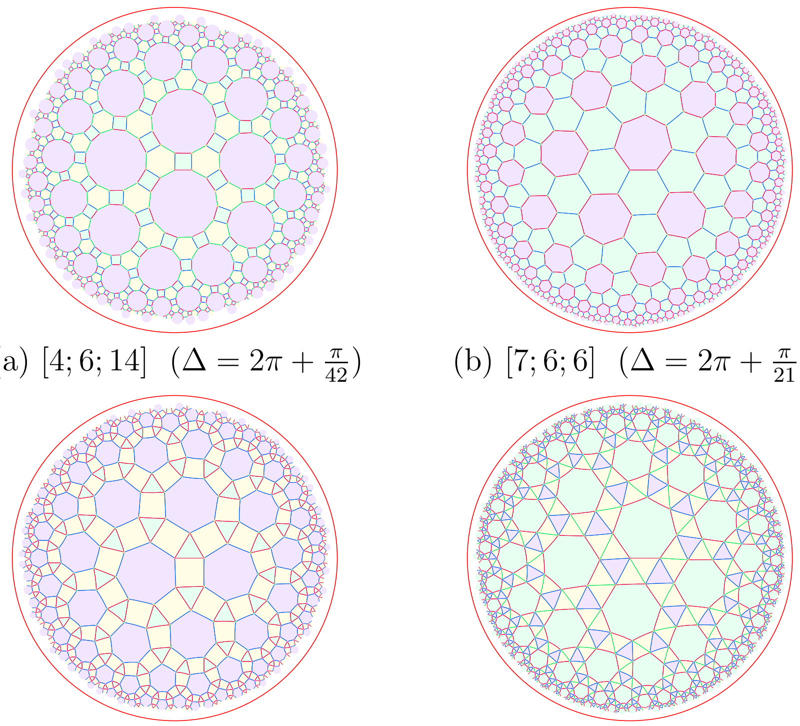
Embeddings of these tilings appear in the figure 9. ━

Example 5 (Enumeration with combinatorial constraints): We have the possibility to enumerate the tilings according to their local properties. For example, we can compute, for a degree d, all the possible type vectors for degree d. For degree 3, this results in $\{[n, n, n], [n, 2m, 2m], [2n, 2m, 2p]\}$ for values of n, m, p such that the elements of the type vectors are greater or equal to 3. For degree 4, we need 4 type vectors, namely:

$$\Big\{[n, m, n, m], [n, 3m, 3m, 3m], [n, 2m, p, 2m], [2n, 2m, 2p, 2q]\Big\}$$

Notice that this list of vectors is strictly larger than the list that can be obtained by enumerating solely the labeling schemes of Cayley graphs [9]. It is also possible, for a given type vector, to decide whether there exist a tiling possessing this type vector and, if this is the case, all the transitive colorings of this tiling. This highlights the fact that a tiling is not uniquely defined by its type vector, and allows for the computation of all its possible labelings preserving the property of uniformity. ━

A final point of interest comes from the fact that the representation is independent of the geometry. The uniform tilings containing infinite faces possess an associated graph of finite treewidth, *i.e.* resembling a tree. Their structure



(a) $[4;6;14]$ $(\Delta = 2\pi + \frac{\pi}{42})$ (b) $[7;6;6]$ $(\Delta = 2\pi + \frac{\pi}{21})$

(c) $[7;4;3;4]$ $(\Delta = 2\pi + \frac{\pi}{21})$ (d) $[7;3;3;3;3]$ $(\Delta = 2\pi + \frac{\pi}{21})$

Fig. 9. Example of tilings of the hyperbolic plane whose Δ is minimal and second minimal among all tilings of the hyperbolic plane.

is relatively simple when compared to the tilings containing only finite faces, which are quasi-isometric to the surface they tile, *i.e.* they cover fairly well this surface. In terms of decidability, these two families are opposite. The fact that it is possible to represent both families with our labeling schemes calls for the devising of geometric algorithms adapted to these kinds of schemes. This has been done for the resolution of particular problems in the case of locally finite Cayley graphs (such as the word problem and the generalized word problem) but could be extended to particular problems for isogonal tilings that are strongly related to the properties of their group of automorphisms.

Thanks. I would like to thank P. Narbel for the quality of his proofreading.

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Number	Opposite vector	Inverse vector	Edge vector	Valid type vectors	Weak ?
1	[1 1 1 1 1]	[0 0 0 0 0]	[1 1 1 1 1]	[n_1;n_1;n_1;n_1;n_1]	
2	[1 1 1 4 4]	[0 0 0 1 0]	[1 1 1 2 2]	[∞;∞;∞;n_4;∞]	Yes
3	[2 1 1 2 5]	[1 1 0 0 0]	[1 1 1 1 2]	[$5n_1$;$5n_1$;$5n_1$;$5n_1$;$5n_1$]	
4	[3 4 1 2 5]	[1 0 1 0 1]	[1 2 1 2 3]	[$10n_1$;$10n_1$;$10n_1$;$10n_1$;$10n_1$]	
5	[3 4 1 2 5]	[0 0 0 0 1]	[1 2 1 2 3]	[$10n_1$;$10n_1$;$10n_1$;$10n_1$;$10n_1$]	
6	[3 4 1 2 5]	[1 0 1 0 0]	[1 2 1 2 3]	[$5n_1$;$5n_1$;$5n_1$;$5n_1$;$5n_1$]	
7	[3 4 1 2 5]	[0 0 0 0 0]	[1 2 1 2 3]	[$5n_1$;$5n_1$;$5n_1$;$5n_1$;$5n_1$]	
8	[4 2 4 1 5]	[1 0 0 1 1]	[1 2 1 1 3]	[$4n_1$;$4n_1$;∞;$4n_1$;∞]	Yes
9	[1 1 3 4 3]	[0 1 0 0 1]	[1 1 2 3 2]	[$5n_1$;$5n_1$;$5n_1$;$5n_1$;$5n_1$]	
10	[2 1 4 3 5]	[1 1 1 1 1]	[1 1 2 2 3]	[$10n_1$;$10n_1$;$10n_1$;$10n_1$;$10n_1$]	
11	[2 1 4 3 5]	[1 1 1 1 0]	[1 1 2 2 3]	[$5n_1$;$5n_1$;$5n_1$;$5n_1$;$5n_1$]	
12	[3 3 1 4 5]	[1 0 1 0 1]	[1 1 1 2 3]	[$5n_1$;∞;$5n_1$;$5n_1$;∞]	Yes
13	[3 2 1 4 5]	[1 0 1 1 0]	[1 2 1 3 4]	[$10n_1$;$10n_1$;$10n_1$;$10n_1$;$10n_1$]	
14	[3 2 1 4 5]	[1 1 1 0 0]	[1 2 1 3 4]	[$10n_1$;$10n_1$;$10n_1$;$10n_1$;$10n_1$]	
15	[3 2 1 4 5]	[1 0 1 0 0]	[1 2 1 3 4]	[$5n_1$;$5n_1$;$5n_1$;$5n_1$;$5n_1$]	
16	[2 1 3 4 5]	[1 1 0 1 0]	[1 1 2 3 4]	[$10n_1$;$10n_1$;$10n_1$;$10n_1$;$10n_1$]	
17	[2 1 3 4 5]	[1 1 1 0 0]	[1 1 2 3 4]	[$10n_1$;$10n_1$;$10n_1$;$10n_1$;$10n_1$]	
18	[2 1 3 4 5]	[1 1 0 0 0]	[1 1 2 3 4]	[$5n_1$;$5n_1$;$5n_1$;$5n_1$;$5n_1$]	
19	[1 2 3 4 5]	[1 0 0 0 0]	[1 2 3 4 5]	[$10n_1$;$10n_1$;$10n_1$;$10n_1$;$10n_1$]	
20	[1 2 3 4 5]	[0 0 0 0 0]	[1 2 3 4 5]	[$5n_1$;$5n_1$;$5n_1$;$5n_1$;$5n_1$]	
21	[2 1 1 2 5]	[0 0 1 1 0]	[1 1 1 1 2]	[n_1;$3n_6$;n_1;$3n_6$;$3n_6$]	
22	[1 2 1 5 5]	[0 0 1 0 1]	[1 2 1 3 3]	[$4n_1$;$4n_1$;$4n_1$;n_2;$4n_1$]	
23	[3 4 1 2 5]	[1 1 1 1 1]	[1 2 1 2 3]	[$2n_1$;$6n_6$;$2n_1$;$6n_6$;$6n_6$]	
24	[3 4 1 2 5]	[1 1 1 1 0]	[1 2 1 2 3]	[$2n_1$;$3n_6$;$2n_1$;$3n_6$;$3n_6$]	
25	[1 1 5 4 5]	[0 1 0 0 1]	[1 1 2 3 2]	[$3n_1$;$3n_1$;$2n_4$;$2n_4$;$3n_1$]	
26	[2 1 4 3 5]	[1 1 0 0 1]	[1 1 2 2 3]	[$8n_1$;$8n_1$;n_3;$8n_1$;$8n_1$]	
27	[2 1 4 3 5]	[1 1 0 0 0]	[1 1 2 2 3]	[$4n_1$;$4n_1$;n_3;$4n_1$;$4n_1$]	
28	[3 3 1 4 5]	[1 0 1 1 1]	[1 1 1 2 3]	[$3n_1$;∞;$3n_1$;$2n_4$;∞]	Yes
29	[4 2 3 1 5]	[0 1 0 0 1]	[1 2 3 1 4]	[$6n_1$;$6n_1$;$6n_1$;$4n_6$;$4n_6$]	
30	[4 2 3 1 5]	[0 0 0 0 1]	[1 2 3 1 4]	[$3n_1$;$3n_1$;$3n_1$;$4n_6$;$4n_6$]	
31	[4 2 3 1 5]	[0 1 0 0 0]	[1 2 3 1 4]	[$6n_1$;$6n_1$;$6n_1$;$2n_4$;$2n_4$]	
32	[4 2 3 1 5]	[0 0 0 0 0]	[1 2 3 1 4]	[$3n_1$;$3n_1$;$3n_1$;$2n_4$;$2n_4$]	
33	[1 3 2 4 5]	[0 0 0 0 1]	[1 2 2 3 4]	[$8n_1$;n_2;$8n_1$;$8n_1$;$8n_1$]	
34	[1 3 2 4 5]	[1 0 0 0 0]	[1 2 2 3 4]	[$8n_1$;n_2;$8n_1$;$8n_1$;$8n_1$]	
35	[1 3 2 4 5]	[0 0 0 0 0]	[1 2 2 3 4]	[$4n_1$;n_2;$4n_1$;$4n_1$;$4n_1$]	
36	[3 2 1 4 5]	[1 1 1 0 1]	[1 2 1 3 4]	[$6n_1$;$4n_6$;$6n_1$;$6n_1$;$4n_6$]	
37	[3 2 1 4 5]	[1 0 1 1 1]	[1 2 1 3 4]	[$8n_1$;$8n_1$;$8n_1$;$2n_4$;$8n_1$]	
38	[2 1 3 4 5]	[1 1 0 1 1]	[1 1 2 3 4]	[$8n_1$;$8n_1$;$8n_1$;$2n_4$;$8n_1$]	
39	[1 2 3 4 5]	[1 0 0 0 1]	[1 2 3 4 5]	[$8n_1$;$8n_1$;$8n_1$;$8n_1$;$2n_6$]	
40	[2 2 5 4 5]	[0 1 0 0 1]	[1 1 2 3 2]	[n_7;$2n_2$;$2n_4$;$2n_4$;$2n_2$]	
41	[2 1 4 3 5]	[0 0 0 0 1]	[1 1 2 2 3]	[n_1;$6n_6$;n_3;$6n_6$;$6n_6$]	
42	[2 1 4 3 5]	[0 0 0 0 0]	[1 1 2 2 3]	[n_1;$3n_2$;n_3;$3n_2$;$3n_2$]	
43	[1 2 4 3 5]	[0 1 0 0 1]	[1 2 3 3 4]	[$4n_1$;$4n_2$;n_3;$4n_2$;$4n_1$]	
44	[1 3 2 4 5]	[1 0 0 0 1]	[1 2 2 3 4]	[$6n_1$;n_2;$6n_1$;$6n_1$;$2n_6$]	
45	[3 2 1 4 5]	[1 1 1 1 1]	[1 2 1 3 4]	[$4n_1$;$4n_6$;$4n_1$;$2n_4$;$4n_6$]	
46	[3 2 1 4 5]	[0 1 0 1 1]	[1 2 1 3 4]	[$4n_1$;$4n_1$;$4n_6$;$2n_4$;$4n_6$]	
47	[3 2 1 4 5]	[0 0 0 1 1]	[1 2 1 3 4]	[$2n_1$;$2n_1$;$4n_6$;$2n_4$;$4n_6$]	
48	[2 1 3 4 5]	[1 1 1 1 1]	[1 1 2 3 4]	[$6n_1$;$6n_1$;$2n_3$;$2n_4$;$6n_1$]	
49	[1 2 3 4 5]	[1 0 0 1 1]	[1 2 3 4 5]	[$6n_1$;$6n_1$;$6n_1$;$2n_4$;$2n_6$]	
50	[2 1 3 4 5]	[0 0 1 1 1]	[1 1 2 3 4]	[n_1;$4n_6$;$2n_3$;$2n_4$;$4n_6$]	
51	[1 2 3 4 5]	[1 0 1 1 1]	[1 2 3 4 5]	[$4n_1$;$4n_1$;$2n_3$;$2n_4$;$2n_6$]	
52	[1 2 3 4 5]	[1 1 1 1 1]	[1 2 3 4 5]	[$2n_1$;$2n_2$;$2n_3$;$2n_4$;$2n_6$]	

Fig. 10. Table containing the 52 labeling schemes of degree 5 representing the possible uniform colored locally finite tilings of degree 5, enumerated by the number of color of edges, separated by their number of colors of faces.

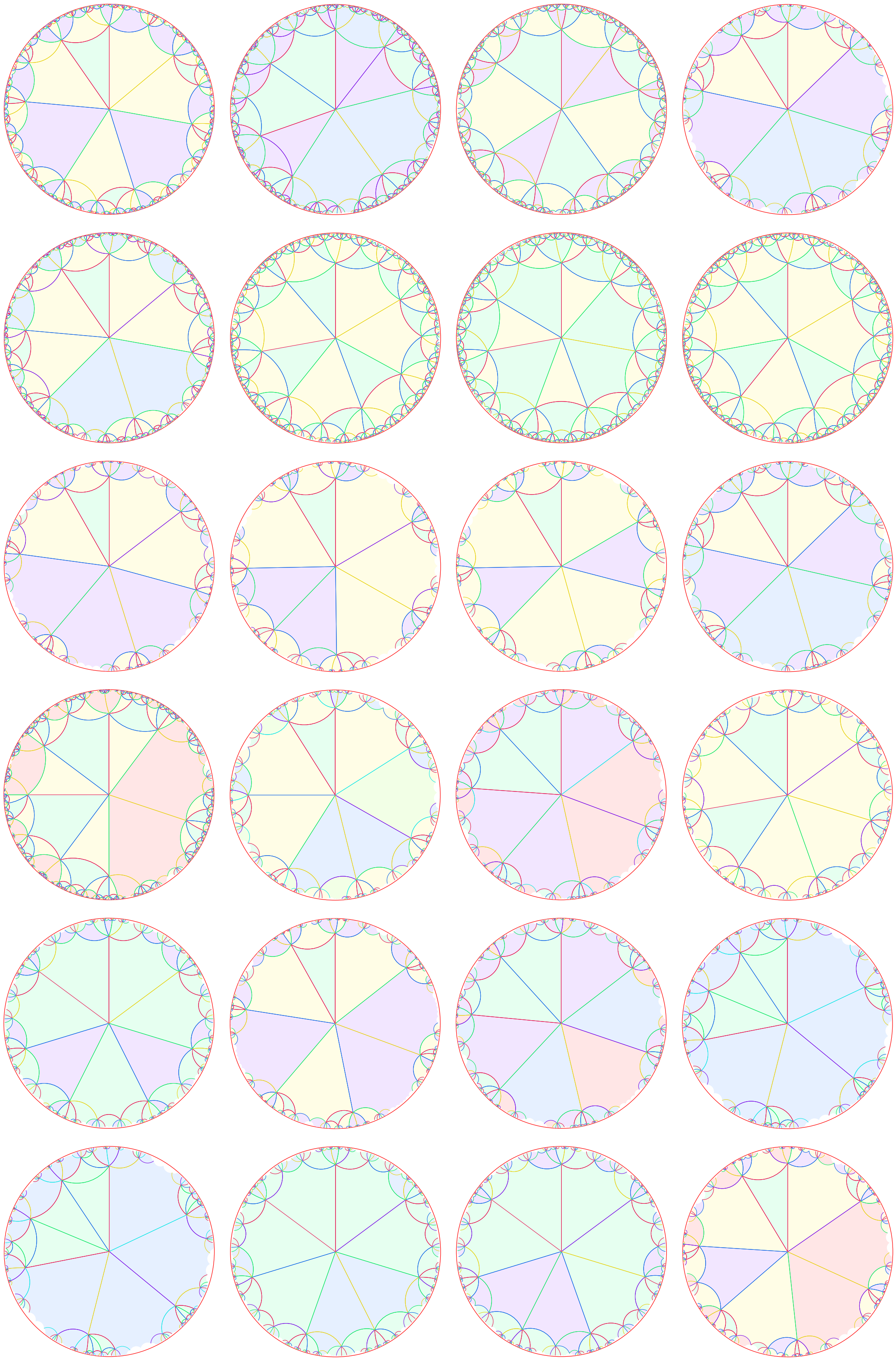

Fig. 11. Table containing 24 embeddings of locally finite planar graphs representing their own labeling scheme, among the 605 possible labeling schemes of degree 7.

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