

Multiple Context-free Grammars

Course 5: MIX is a 2-MCFL

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Outline

The *MIX* problem

MIX as a group language

A grammar for O_2

Proof of the Theorem

A Theorem on Jordan curves

Conclusion on MIX

On MCFLs

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MIX

$$MIX = \{w \in \{a; b; c\}^* \mid |w|_a = |w|_b = |w|_c\}$$

The Bach language

► Bach (1981)



Exercise 2: Let $L = \{X \mid X = (abc)^n\}$. L is CF (in fact regular).
 But Scramble (L) is not CF. For let $L' = \{X \mid X = a^n b^m c^k\}$ then
 $L' \cap L = \{X \mid X = a^n b^n c^n\}$ is not CF, but since the intersection of
 a CF language and a regular language is CF, L can't be CF.

Wikipedia entry:

http://en.wikipedia.org/wiki/Bach_language

The *MIX* language

- ▶ Marsh (1985)

Conjecture: *MIX* is not an indexed language.

Proof. Consider the language $MIX = SCRAMBLE((abc)^+)$ (the names 'mix' and 'MIX' — pronounced 'little mix' and 'big mix' were the happy invention of Bill Marsh; 'little mix' is the scramble of $(ab)^+$).

MIX and Tree Adjoining Grammars



► Joshi (1985)

[*MIX*] represents the extreme case of the degree of free word order permitted in a language. This extreme case is linguistically not relevant. [...] TAGs also cannot generate this language although for TAGs the proof is not in hand yet.

MIX and Tree Adjoining Grammars

- ▶ Vijay Shanker, Weir, Joshi (1991)



of strings of equal number of a 's, b 's, and c 's in any order. MIX can be regarded as the extreme case of free word order. It is not known yet whether TAG, HG, CCG and LIG can generate MIX. This has turned out to be a very difficult problem. In fact, it is not even known whether an IG can generate MIX.

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Group languages

Group finite presentation:

- ▶ a finite set of generators Σ
- ▶ a finite set of defining equations E

Word problem: given w in Σ^* , is $w =_E 1$?

Group language: $\{w \in \Sigma^* \mid w =_E 1\}$

- ▶ the word problem is in general undecidable (Novikov 1955, Boone 1958)
- ▶ the languages of different representation of a group are rationally equivalent
- ▶ relate algebraic properties of groups to language-theoretic properties of their group languages

Example: a group language is context free iff its underlying group is virtually free (Muller Schupp 1983)

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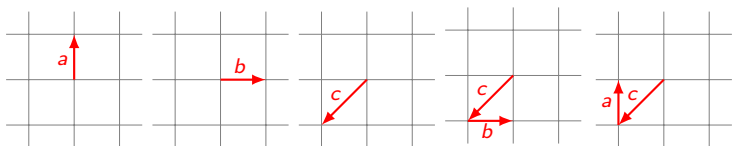
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MIX as a group language

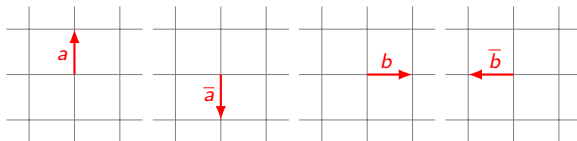
- ▶ Generators: $\{a; b; c\}$
- ▶ Defining equations: $a^{-1} = bc = cb$, $b^{-1} = ac = ca$,
 $c^{-1} = ab = ba$

\mathbb{Z}^2 is the group that has this presentation.



Yet another presentation of Z^2

- Generators: $\{a; \bar{a}; b; \bar{b}\}$
- Defining equations: $a^{-1} = \bar{a}$, $b^{-1} = \bar{b}$, $ab = ba$, $a\bar{b} = \bar{b}a$, $\bar{a}b = b\bar{a}$, $\bar{a}\bar{b} = \bar{b}\bar{a}$



The associated group language is

$$O_2 = \{w \in \{a; \bar{a}; b; \bar{b}\}^* \mid |w|_a = |w|_{\bar{a}} \wedge |w|_b = |w|_{\bar{b}}\}$$

MIX and O_2 are rationally equivalent

The following transductions are due to Kanazawa:

- ▶ There is a rational transduction from O_2 to MIX :
let $R = \{a|b|\bar{a}\bar{b}\}^*$, then $MIX = h(O_2 \cap R)$ if $h(a) = a$,
 $h(b) = b$, $h(\bar{a}) = c$ and $h(\bar{b}) = \epsilon$.
- ▶ There is a rational transduction from MIX to O_2 :
let $R = \{abab|cc|cbcb|aa\}^*$, then O_2 , then
 $O_2 = g^{-1}(MIX \cap R)$ with $g(a) = abab$, $g(\bar{a}) = cc$,
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NB: $w \in MIX \cap R$ iff $|w|_{abab} + |w|_{aa} = |w|_{cbcb} +$
 $|w|_{abab} = |w|_{cc} + |w|_{cbcb}$ iff $|w|_{abab} = |w|_{cc}$ and
 $|w|_{cbcb} = |w|_{aa}$.

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- ▶ Thus *MIX* belongs to a rational cone iff O_2 does.
- ▶ Thus *MIX* is a 2-MCFL iff O_2 is a 2-MCFL.

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let $R = \{abab|cc|cbcb|aa\}^*$, then O_2 , then
 $O_2 = g^{-1}(MIX \cap R)$ with $g(a) = abab$, $g(\bar{a}) = cc$,
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- ▶ Thus MIX belongs to a rational cone iff O_2 does.
- ▶ Thus MIX is a 2-MCFL iff O_2 is a 2-MCFL.

MIX and computational group theory

- ▶ Gilman (2005)



is indexed but not context free seems to have been open for several years. It does not even seem to be known whether or not the word problem of $Z \times Z$ is indexed.

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A 2-MCFG for O_2

$$\begin{array}{c}
 S(xy) \leftarrow Inv(x, y) \\
 \hline
 Inv(x_1 y_1, y_2 x_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2) \\
 Inv(x_1 x_2 y_1, y_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2) \\
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 Inv(\alpha x_1 \bar{\alpha}, x_2) \leftarrow Inv(x_1, x_2) \\
 Inv(\alpha x_1, \bar{\alpha} x_2) \leftarrow Inv(x_1, x_2) \\
 Inv(\alpha x_1, x_2 \bar{\alpha}) \leftarrow Inv(x_1, x_2) \\
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 Inv(x_1, \alpha x_2 \bar{\alpha}) \leftarrow Inv(x_1, x_2) \\
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 Inv(x_1 y_1 x_2, y_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2) \\
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 \hline
 Inv(\epsilon, \epsilon) \leftarrow
 \end{array}$$

where $\alpha \in \{a; b\}$

Theorem: Given w_1 and w_2 such that $w_1 w_2 \in O_2$, $Inv(w_1, w_2)$ is derivable.

A 2-MCFG for O_2

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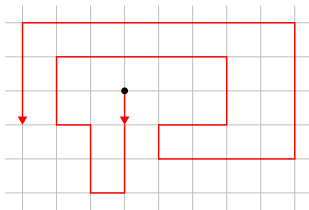
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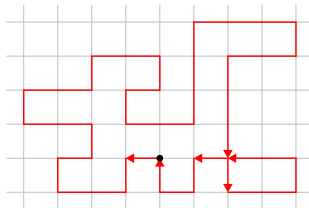
Graphical interpretation of the word $\overline{aaabaabaabbbbbb\overline{aabb\overline{abbbbaaaabbbbbb\overline{bbbaa}}}}$:



Graphical interpretation of the word $\overline{aa}\overline{ab}a\overline{a}ba\overline{abb}bbbbb\overline{a}\overline{abb}\overline{b}bbbbaaa\overline{ab}bbbbbbbbb\overline{aaa}$:

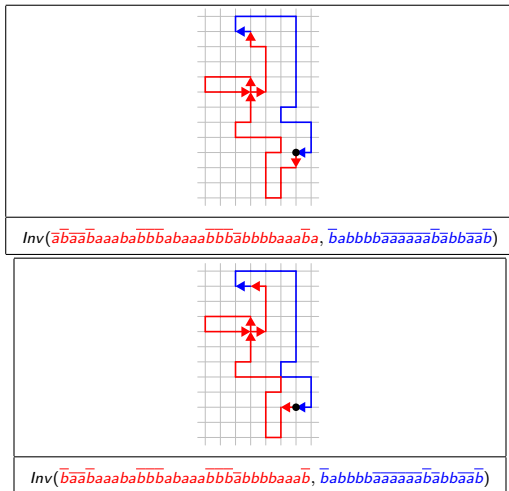


The words in O_2 are precisely the words that are represented as closed curves:
 $\bar{b}\bar{a}\bar{b}\bar{b}ababb\bar{b}abbabb\bar{a}\bar{b}\bar{a}\bar{b}baaa\bar{b}\bar{b}\bar{b}\bar{a}\bar{b}\bar{b}\bar{a}\bar{a}\bar{a}\bar{a}\bar{b}\bar{b}abbb\bar{b}\bar{a}$



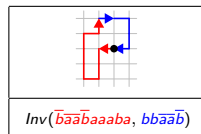
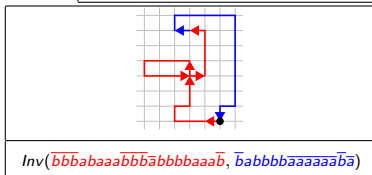
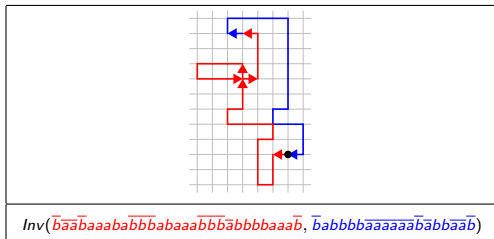
Parsing with the grammar

Rule $Inv(\bar{a}x_1a, x_2) \leftarrow Inv(x_1, x_2)$



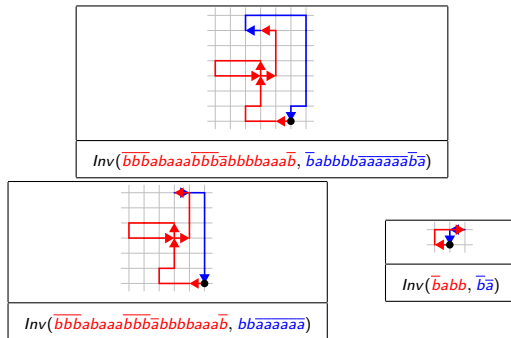
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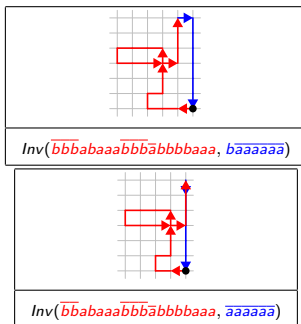


Rule: $Inv(x_1 \bar{b}, bx_2) \leftarrow Inv(x_1, x_2)$



Parsing with the grammar

Rule: $Inv(\overline{b}x_1, bx_2) \leftarrow Inv(x_1, x_2)$



Rule: $Inv(x_1y_1, y_2x_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2)$



Rule: $Inv(\bar{b}x_1b, x_2) \leftarrow Inv(x_1, x_2)$



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The proof of the Theorem

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The proof is done by induction on the lexicographically ordered pairs $(|w_1 w_2|, \max(|w_1|, |w_2|))$.

There are five cases:

Case 1: w_1 or w_2 equal ϵ :

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Case 1: w_1 or w_2 equal ϵ :

w.l.o.g., $w_1 \neq \epsilon$, then by induction hypothesis, for any v_1 and v_2 different from ϵ such that $w_1 = v_1 v_2$, $Inv(v_1, v_2)$ is derivable then:

$$\frac{Inv(v_1, v_2) \quad Inv(\epsilon, \epsilon)}{Inv(v_1 v_2 = w_1, \epsilon)} \quad Inv(x_1 x_2 y_1, y_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2)$$

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There are five cases:

Case 2: $w_1 = s_1 w'_1 s_2$ and $w_2 = s_3 w'_2 s_4$ and for $i, j \in \{1; 2; 3; 4\}$, s.t. $i \neq j$, $\{s_i; s_j\} \in \{\{a; \bar{a}\}; \{b; \bar{b}\}\}$:

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e.g., if $i = 1, j = 2$, $s_1 = a$ and $s_2 = \bar{a}$ then by induction hypothesis $Inv(w'_1, w_2)$ is derivable and:

$$\frac{Inv(w'_1, w_2)}{Inv(aw'_1\bar{a}, w_2)} \quad Inv(ax_1\bar{a}, x_2) \leftarrow Inv(x_1, x_2)$$

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Case 3: the curves representing w_1 and w_2 have a non-trivial intersection point:

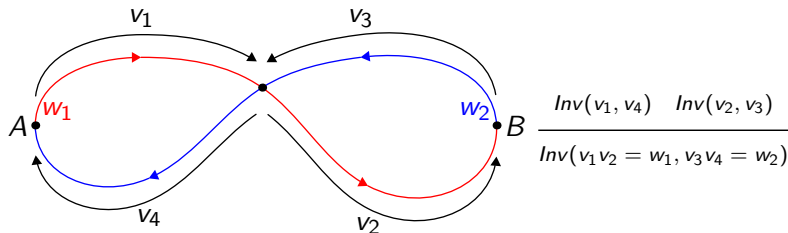
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There are five cases:

Case 4: the curve representing w_1 or w_2 starts or ends with a loop:

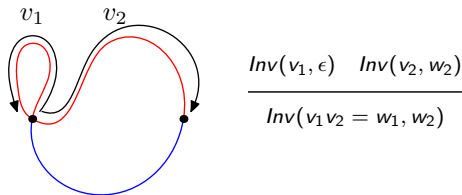
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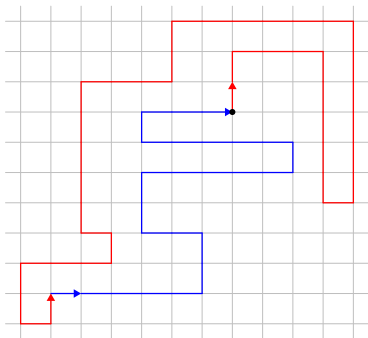
There are five cases:

Case 5: w_1 and w_2 do not start or end with compatible letters, the curve representing then do not intersect and do not start or end with a loop.

No other rule than

$$Inv(x_1, y_1 x_2 y_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2)$$

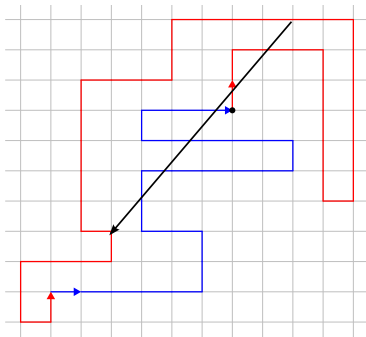
can be used.



No other rule than

$$Inv(x_1, y_1 x_2 y_2) \leftarrow Inv(x_1, x_2), Inv(y_1, y_2)$$

can be used.



The relevance of case 5

The word

$abb\overline{a\overline{a}baa}\overline{abbb\overline{b}aa}aba$

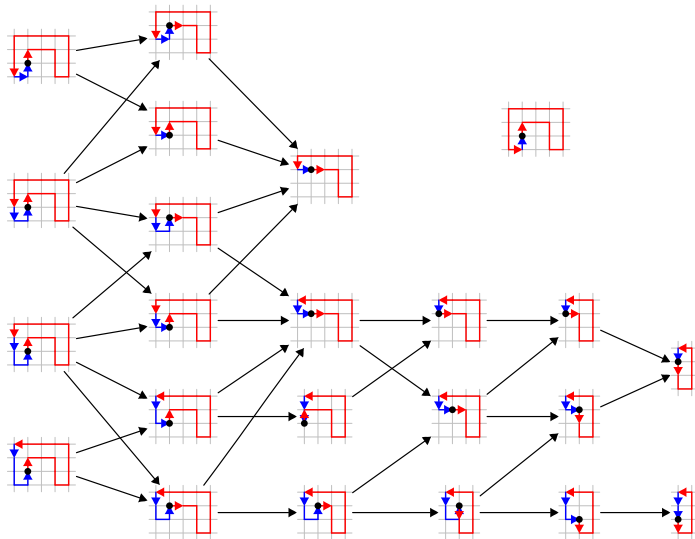


is not in the language of the grammar only containing the non-well-nested rules.

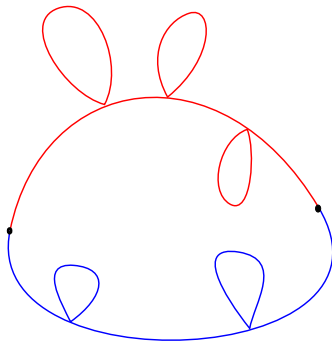
Multiple Context-free Grammars

└ Proof of the Theorem

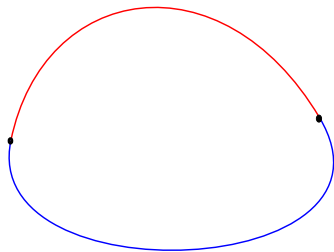




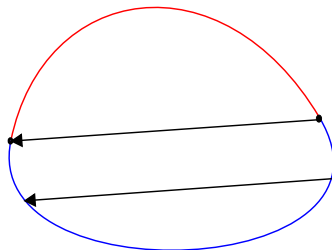
Solving case 5: towards geometry



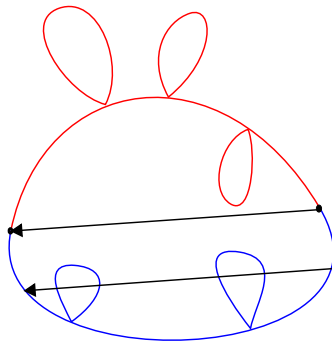
Solving case 5: towards geometry



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Solving case 5: towards geometry

- ▶ w.l.o.g. we may assume that w_1 and w_2 start and end with a or b ,
- ▶ if we consider subwords w'_1 and w'_2 of w_1 and w_2 obtained by erasing factors of w_1 and w_2 that are in O_2 , we have:
 - ▶ w'_1 and w'_2 start or end with a or b ,
 - ▶ the curve that represents $w'_1 w'_2$ is a Jordan curve,
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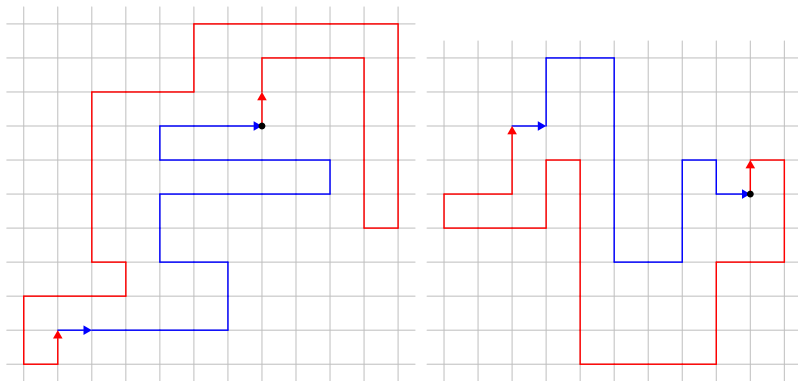
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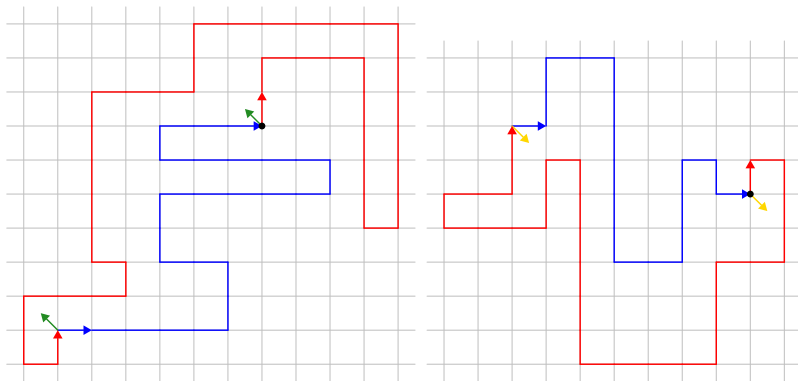
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Solving case 5: a geometrical invariant

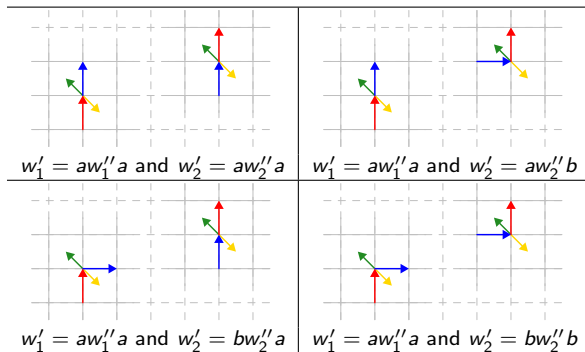


Solving case 5: a geometrical invariant



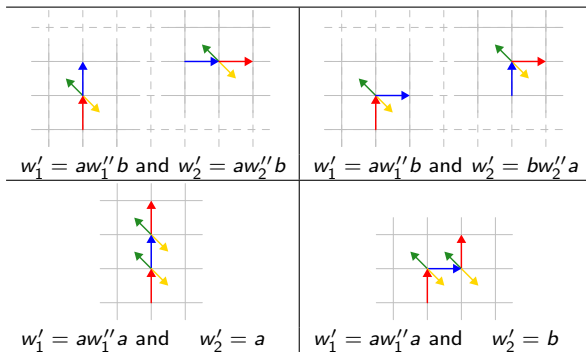
Solving case 5: a geometrical invariant

An invariant on the Jordan curve representing $w'_1 w'_2$:



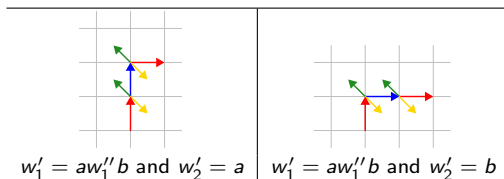
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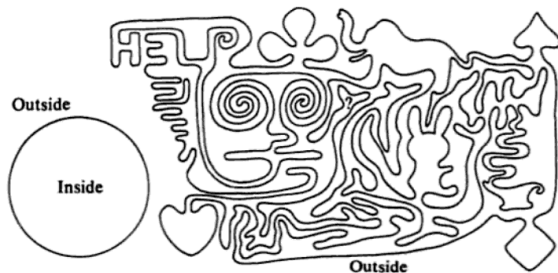


Figure 13.1 Two Jordan curves.

illustration from: A combinatorial introduction to topology by Michael Henle (Dover Publications).

Theorem: There is $k \in \{-1; 1\}$ such that the winding number of Jordan curve around a point in its interior is k , its winding number around a point in its exterior is 0.

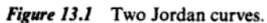
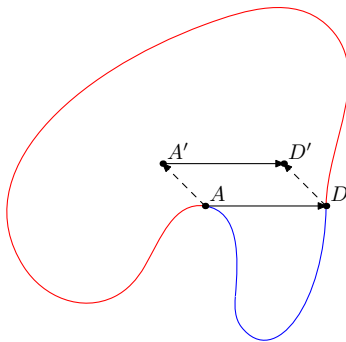


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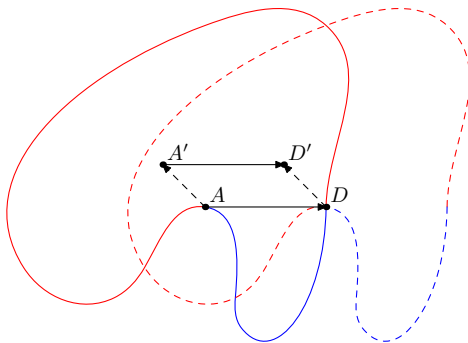
A theorem on Jordan curves

Theorem: If A and D are two points on a Jordan curve J such that there are two points A' and D' inside J such that $\overrightarrow{AD} = \overrightarrow{A'D'}$, then there are two points B and C pairwise distinct from A and D such that A, B, C , and D appear in that order on one of the arcs going from A to D and $\overrightarrow{AD} = \overrightarrow{BC}$.



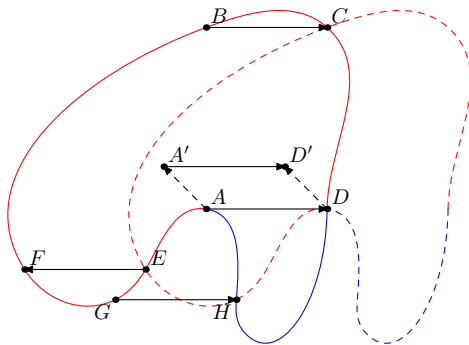
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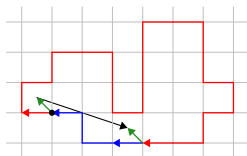
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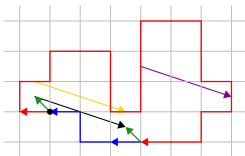
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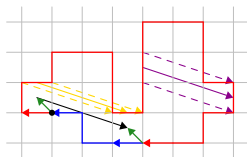
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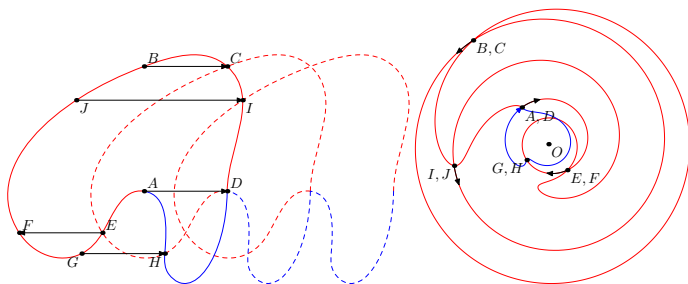
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Simple curves, translations, intersections and the complex exponential

Let's suppose that $D - A = 1$

$$\text{let } \varphi : \begin{cases} \mathbb{C} & \rightarrow \mathbb{C} - \{0\} \\ z & \rightarrow e^{2i\pi z} \end{cases}.$$

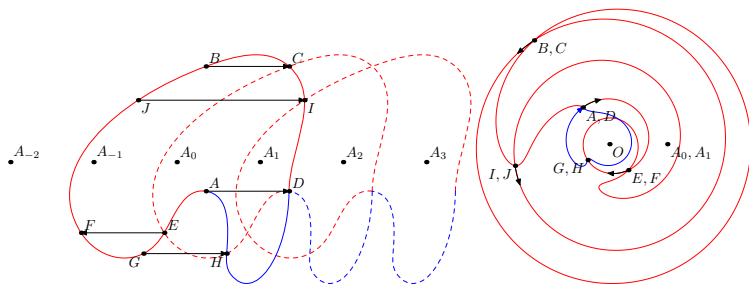


φ transforms arcs performing translation of k into arc that have k as winding number around 0.

Simple curves, translations, intersections and the complex exponential

Let's suppose that $D - A = 1$ and that $A_0 = A' = 0$, $A_1 = D' = 1, \dots$, $A_k = k$

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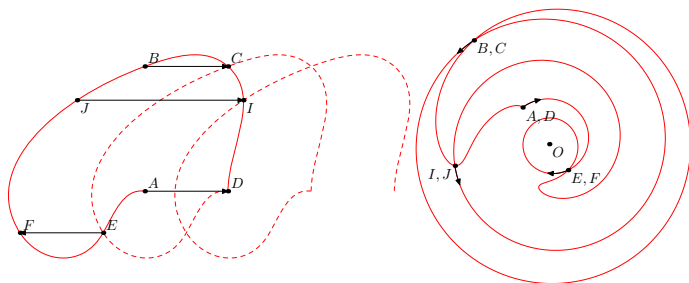


φ sums up the winding number of a Jordan curve around the A_i 's as the winding number around $\varphi(A_0) = \varphi(0) = 1$.

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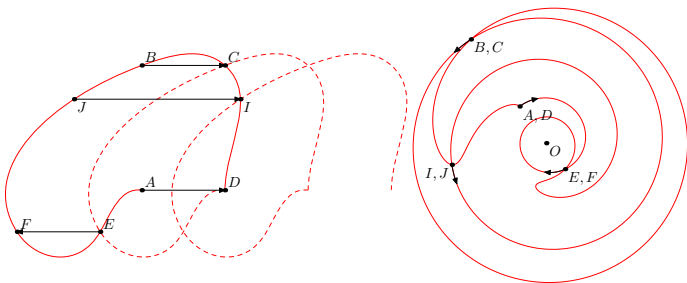
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Main Lemma: a simple arc J from A to D (resp. D to A) does not contain B and C as required in the Theorem iff $\varphi(J)$ is a Jordan curve of $\mathbb{C} - \{0\}$ that belong to the homotopy class 1 (resp. -1).

Proof of the main Lemma

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Proof.

Let $wn(J, z)$ be the winding number of a closed curve around z .

- ▶ If $\varphi(J)$ contains a closed subarc J' with $wn(J, 0) = k$, then it corresponds to a subarc of J going from a point E to $E + k$,
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Corollary: a simple path J from A to D (*resp.* D to A) does not contain B and C as required in the Theorem iff $\varphi(J)$ is a Jordan curve of $\mathbb{C} - \{1\}$ that belong to the homotopy class 0 or 1 (*resp.* or -1).

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Corollary: if J is a simple closed curve of \mathbb{C} composed with two curves J_1 and J_2 respectively going from A to D and D to A which do not contain points B and C as required in the Theorem then $|wn(\varphi(J), 1)| = |wn(\varphi(J_1), 1) + wn(\varphi(J_2), 1)| \leq 1$.

Lemma: if J is a simple closed curve of \mathbb{C} composed with two curves J_1 and J_2 respectively going from A to D and D to A such that 0 and 1 are in the interior of J , then either $|wn(\varphi(J), 1)| \geq 2$.

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- ▶ we have showed that O_2 is a 2-MCFL exhibiting the first non-virtually free group language that is proved to belong to an interesting class of language,
- ▶ this implies that contrary to the usual conjecture we have showed that *MIX* is a 2-MCFLs.

Conjectures

Well-nestedness:

$$\frac{\text{Well-nested} \quad \text{Inv}(y_1 x_1 x_2, y_2) \leftarrow \text{Inv}(x_1, x_2), \text{Inv}(y_1, y_2)}{\text{Not well-nested} \quad \text{Inv}(y_1 x_1 y_2, x_2) \leftarrow \text{Inv}(x_1, x_2), \text{Inv}(y_1, y_2)}$$

MCFG_{wn} are MCFGs with well-nested rules.

- ▶ MCFL_{wn} coincide with non-duplicating IO/OI,
- ▶ MCFL is incomparable with IO or OI.

Thus the following conjectures:

- ▶ mildly context sensitive languages may well be, as advocated by Kanazawa, MCFL_{wn}
- ▶ O_2 and MIX should not be a MCFL_{wn}
- ▶ semilinear rational cones included in OI should be included in MCFL_{wn}
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Open question:

- ▶ Is O_3 an MCFL?

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Conjectures

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MCFG_{wn} are MCFGs with well-nested rules.

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Thus the following conjectures:

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Open question:

- ▶ Is O_3 an MCFL?

Outline

The *MIX* problem

MIX as a group language

A grammar for O_2

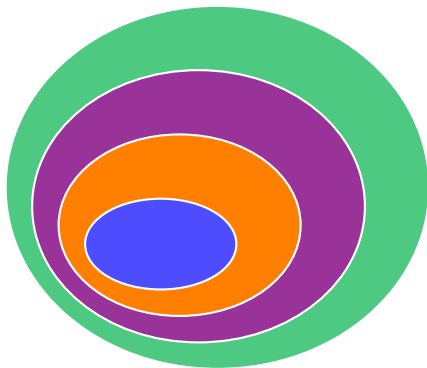
Proof of the Theorem

A Theorem on Jordan curves

Conclusion on MIX

On MCFLs

MCFL a natural class



$$\begin{aligned}
 MCTAG &= MCFG = HR \\
 &= OUT(DTWT) \\
 &= yDT_{fc}(REGT) = LUSCG = MG \\
 &= \lambda\text{-CFL}_{lin}(4) = \lambda\text{-CFL}_{lin}
 \end{aligned}$$

$$\begin{aligned}
 MCFG_{wn} &= IO_{nd} = OI_{nd} \\
 &= CCFG = \lambda\text{-CFL}_{lin}(3)
 \end{aligned}$$

$$\begin{aligned}
 TAG &= LIG = CCG = HG \\
 &= 2\text{-}MCFG_{wn}
 \end{aligned}$$

$$CFL = 1\text{-}MCFL = \lambda\text{-CFL}_{lin}(2)$$

Yet an counter-intuitive class

- ▶ Joshi, Vijay Shanker, Weir (1991)



in MCSL; 2) languages in MCSL can be parsed in polynomial time; 3) MCSGs capture only certain kinds of dependencies, e.g., nested dependencies and certain limited kinds of crossing dependencies (e.g., in the subordinate clause constructions in Dutch or some variations of them, but perhaps not in the so-called MIX (or Bach) language, which consists of equal numbers of a's, b's, and c's in any order 4) languages in MCSL have constant growth property, i.e., if the strings of a language

MIX is a 2-MCFL(2)

Yet another counter-intuitive class

- Groenink (PhD dissertation): *finite pumpability*



Definition: Finite pumpability. A language L is FINITELY PUMPABLE if there is a constant c_0 such that for any $w \in L$ with $|w| > c_0$, there are a finite number k and strings u_0, \dots, u_k and v_1, \dots, v_k such that $w = u_0 v_1 u_1 v_2 u_2 \cdots u_{k-1} v_k u_k$, and for each i , $1 \leq |v_i| < c_0$, and for any $p \geq 0$, $u_0 v_1^p u_1 v_2^p u_2 \cdots u_{k-1} v_k^p u_k \in L$.

There is a 3-MCFL which is not finitely pumpable.

Are MCFLs interesting for natural languages?

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- ▶ They pose challenging problems
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The last word...

- ▶ Chomsky (2004)



pened. The systems that capture other properties of language, for example transformational grammar, hold no interest for mathematics. But I do not think that that is a necessary truth. It could turn out that there would be richer or more appropriate mathematical ideas that would capture other, maybe deeper properties of language than context-free grammars do. In that case you have another branch of applied mathematics which might have linguistic consequences. That could be exciting.