Morphic words and recursion schemes

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FREC meeting, LABRI

6 dec 2011

Context

Graphs and trees with decidable MSO-theory :

- pushdown/Caucal hierarchy
- recursion schemes : [Damm], and recently [Knapik-Niwiński-Urzyczyn], [Ong]
- infinite words : ult. periodic, morphic [Carton-Thomas]

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This talk :

1. relationship between order-1 schemes and morphic words
2. extension to order 2

Recursion schemes : first order

Term grammar with

- terminals $T = \{a, b, f, g \dots\}$,
- nonterminals $N = \{S, F, G \dots\}$,
- a specific starting nonterminal S
- one rewriting rule per nonterminal, using variables $\mathcal{X} = \{x, y \dots\}$.

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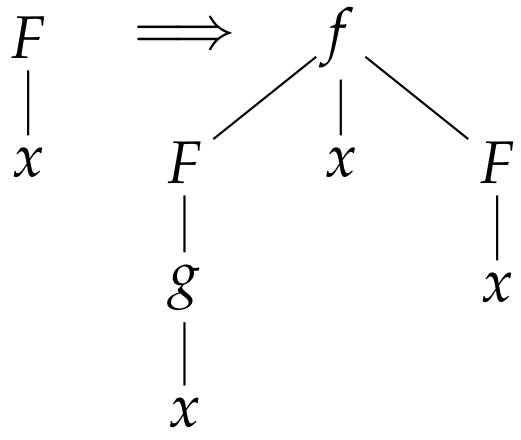
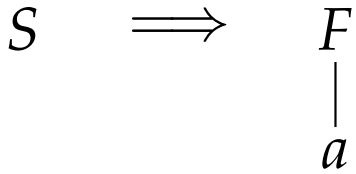
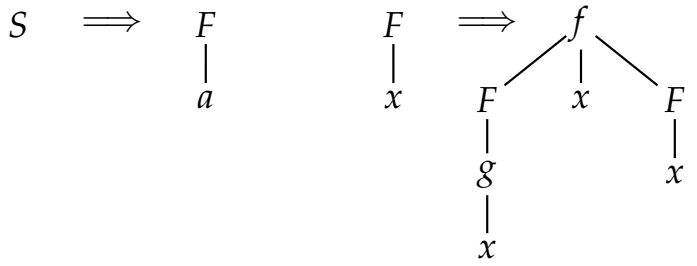
$$N = \{S, F\}$$



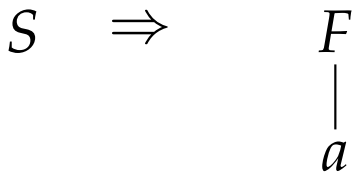


Here $\rho(f) = 3, \rho(F) = \rho(g) = 1, \rho(a) = 0$.

Recursion schemes : first order



A scheme builds a (possibly) infinite tree of terminals.



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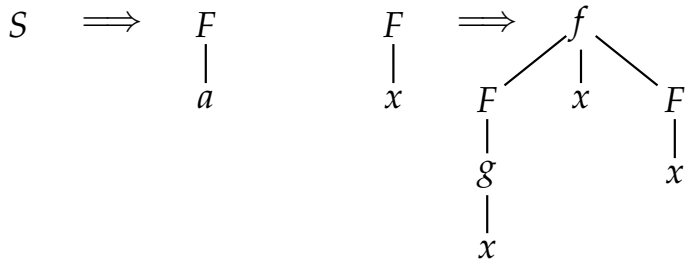

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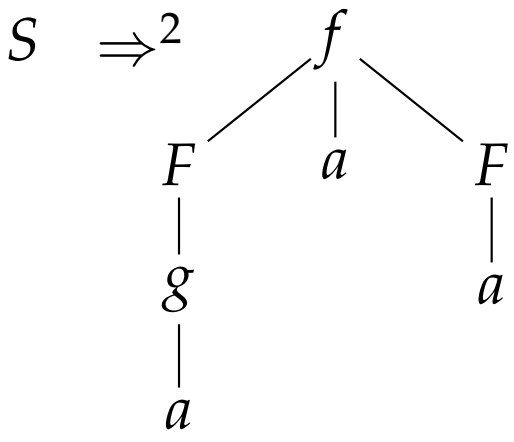

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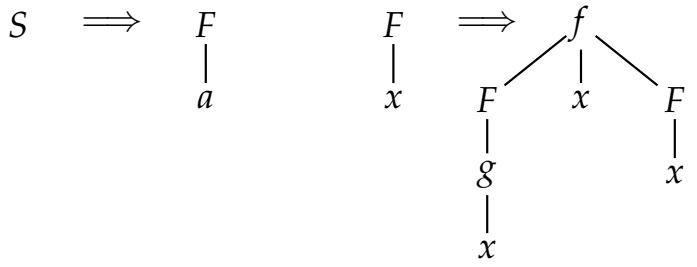

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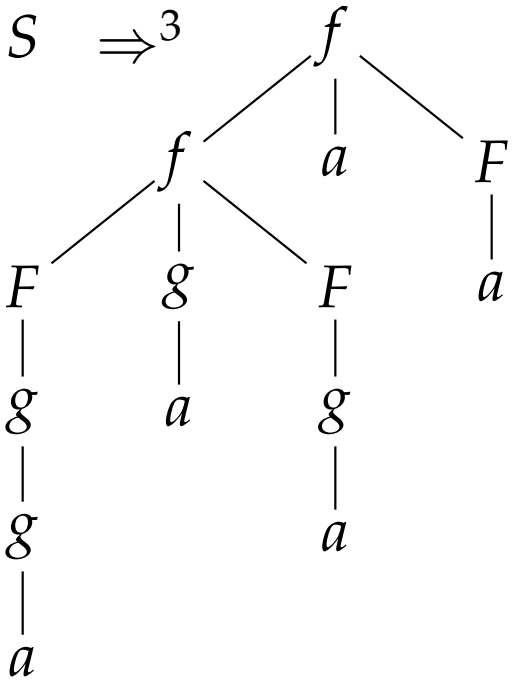

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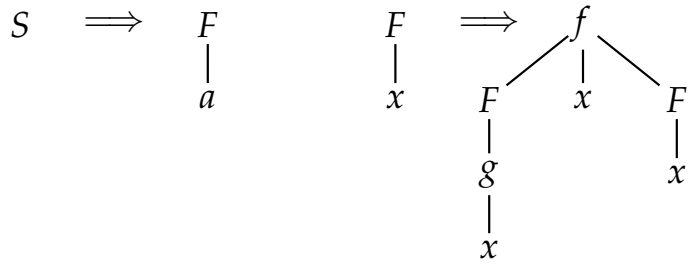

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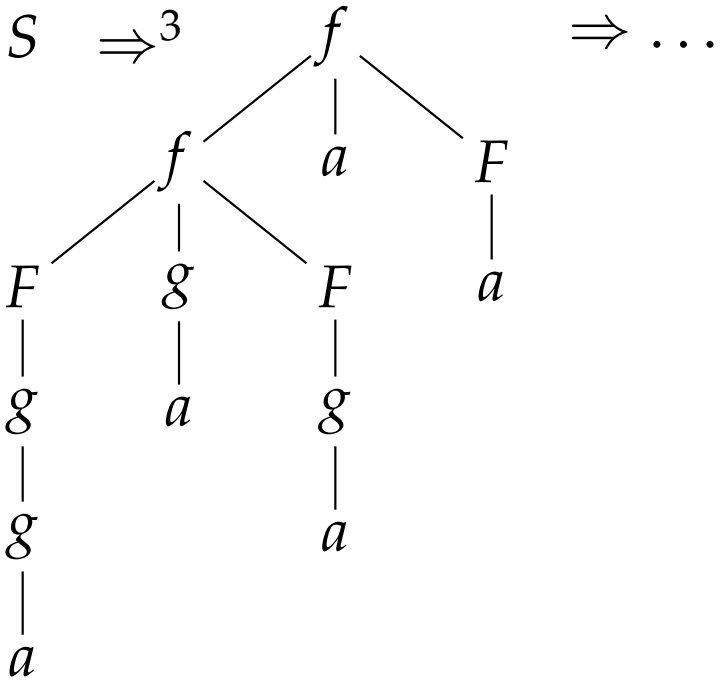

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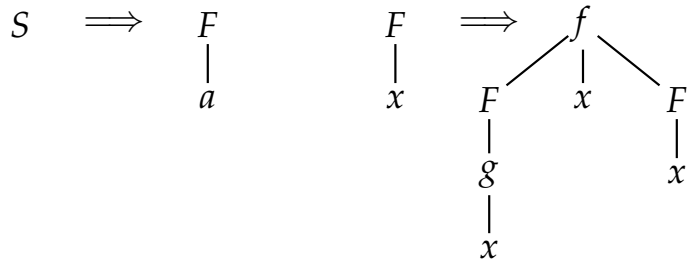

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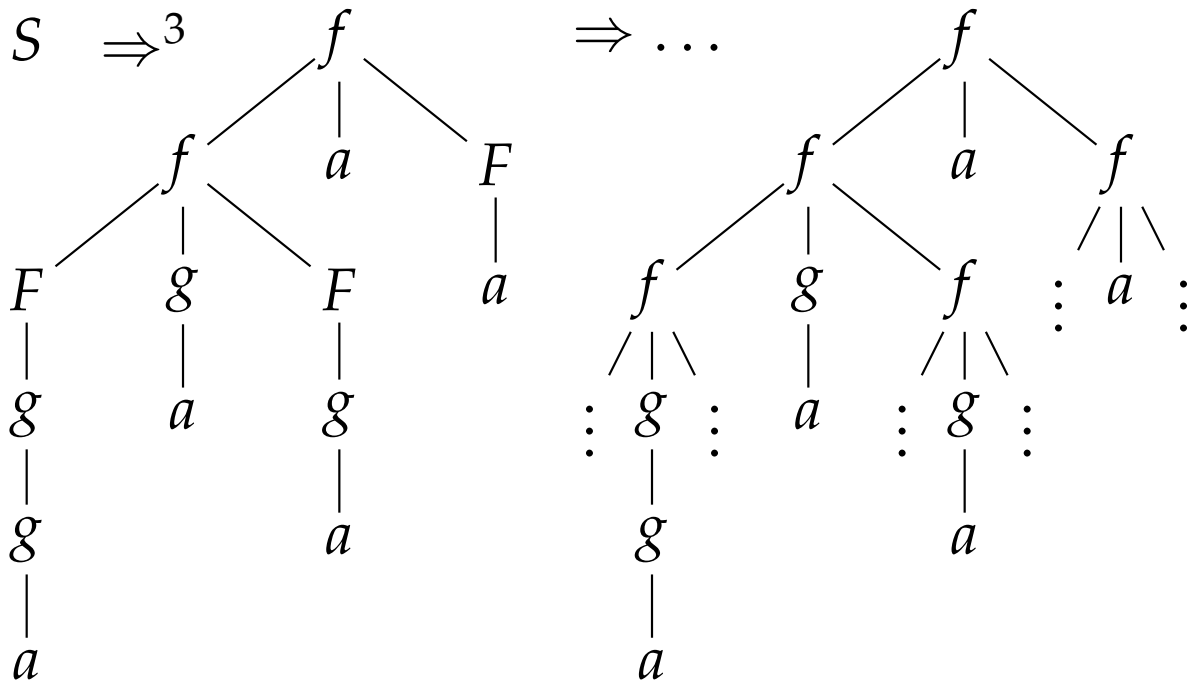
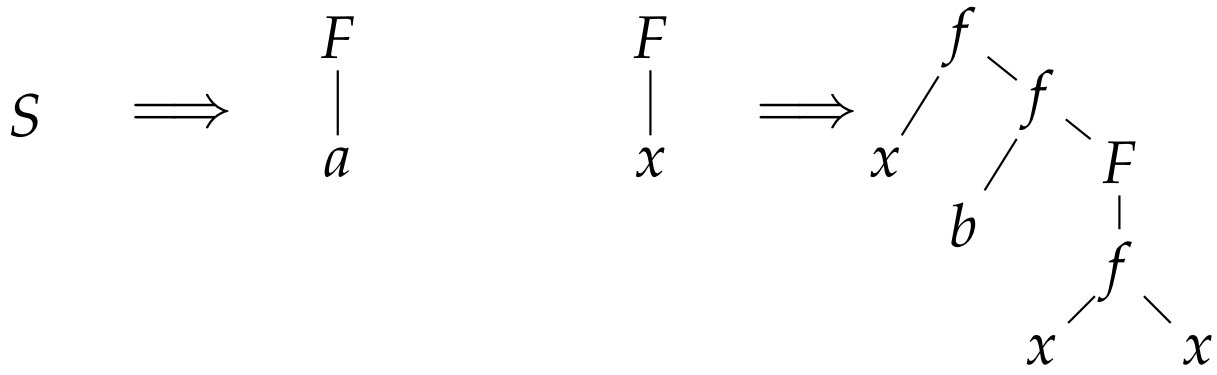
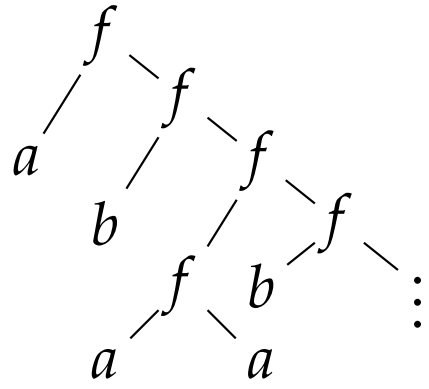


Frontiers of limit trees

We are interested in infinite words that appear in schemes.



$[\![S]\!] =$



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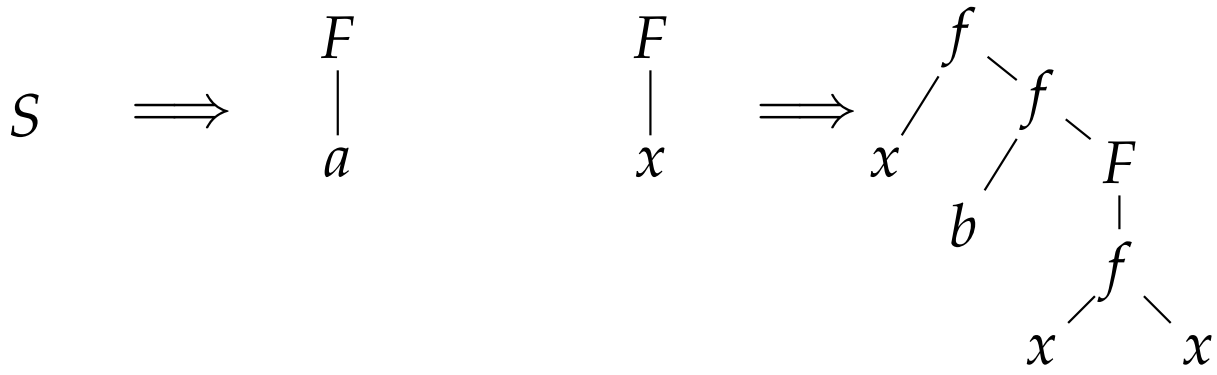

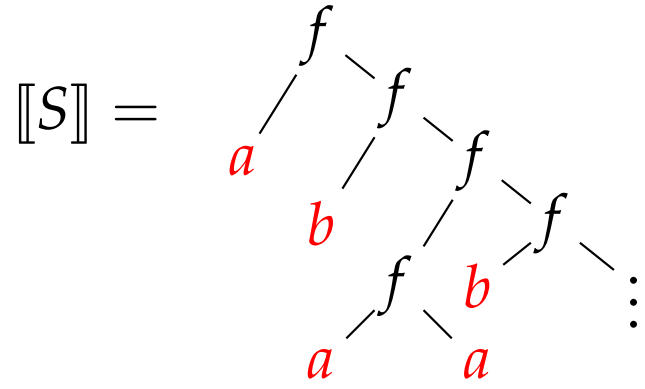

$$\mathbf{Fr}(\llbracket S \rrbracket) = abaab \ldots a^{2^i} b \ldots$$

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Proposition

For any tree generated by an order-1 scheme, there is a tree generated by a order-1 scheme where the rightmost branch is the only infinite branch, and with the same ω-frontier.

The trees with one infinite rightmost branch are called *combs*.

Morphic words

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Let τ be a morphism on Σ^* s.t. there is $a \in \Sigma$ with $\tau(a) \in a\Sigma$.

$$\begin{aligned} \tau(a) &= au \\ \tau^2(a) &= au\tau(u) \\ \tau^3(a) &= au\tau(u)\tau^2(u) \\ &\dots \\ \tau^\omega(a) &= au\tau(u)\dots \end{aligned}$$

Words $\sigma(\tau^\omega(a))$ are *morphic words*, where σ is another morphism.

Morphic words : example

$$\begin{aligned} \tau(a) &= abcc \\ \tau(b) &= b \\ \tau(c) &= cc \end{aligned} \qquad \begin{aligned} \sigma(a) &= a \\ \sigma(b) &= b \\ \sigma(c) &= a \end{aligned}$$

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First result

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For any order-1 scheme, there is an order-1 scheme generating the same tree and where every nonterminal has only useful parameters.

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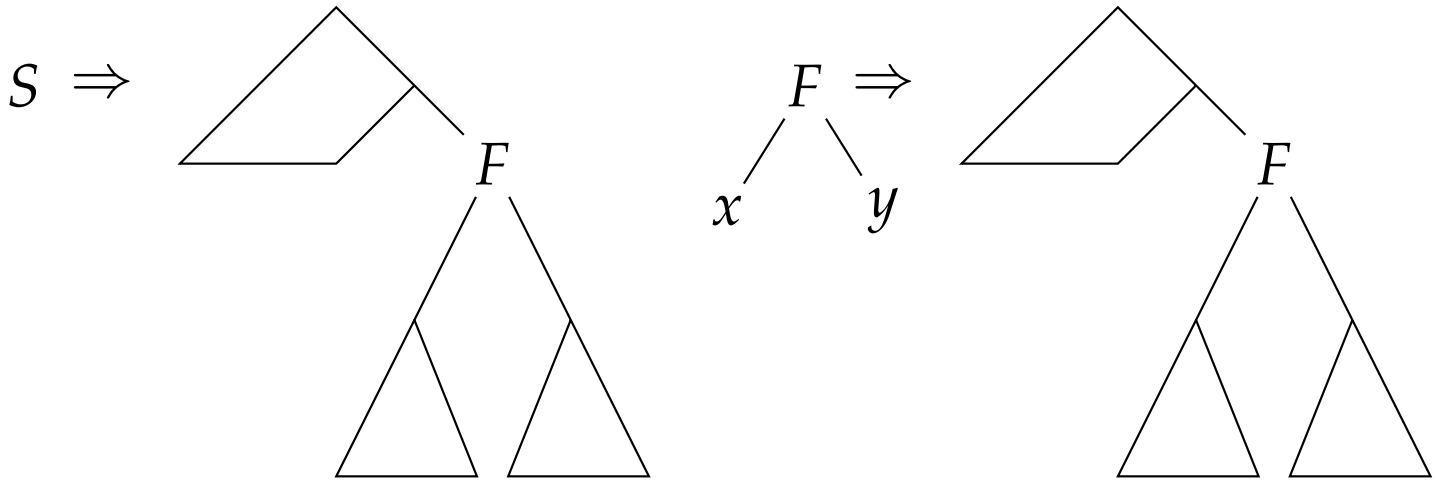
Lemma (usefulness)

For any order-1 scheme, there is an order-1 scheme generating the same tree and where every nonterminal has only useful parameters.

Lemma (linearization)

For any order-1 scheme $\mathfrak{S}$ generating a comb, there is $\mathfrak{S}'$ with only two nonterminals $\{S, R\}$ such that $\mathbf{Fr}(\mathfrak{S}) = \mathbf{Fr}(\mathfrak{S}')$. Moreover, each rewriting rule has exactly one occurence of R and none of S.

Proof sketch



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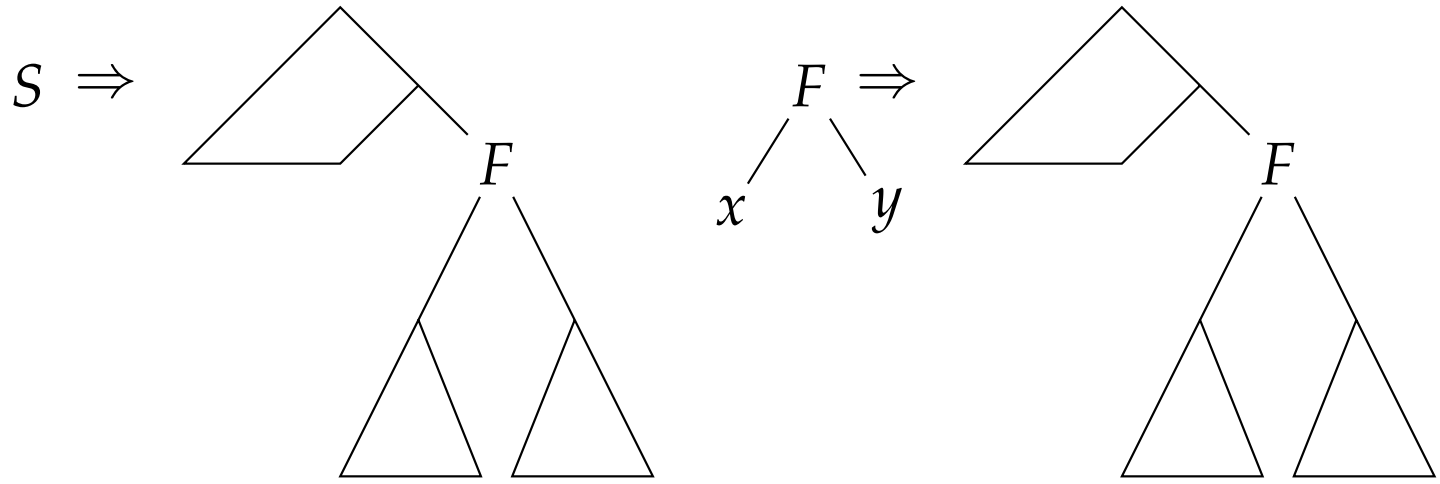

- Letters : $\{c \in T \mid \rho(c) = 0\} \cup \{x, y, \Delta\}$
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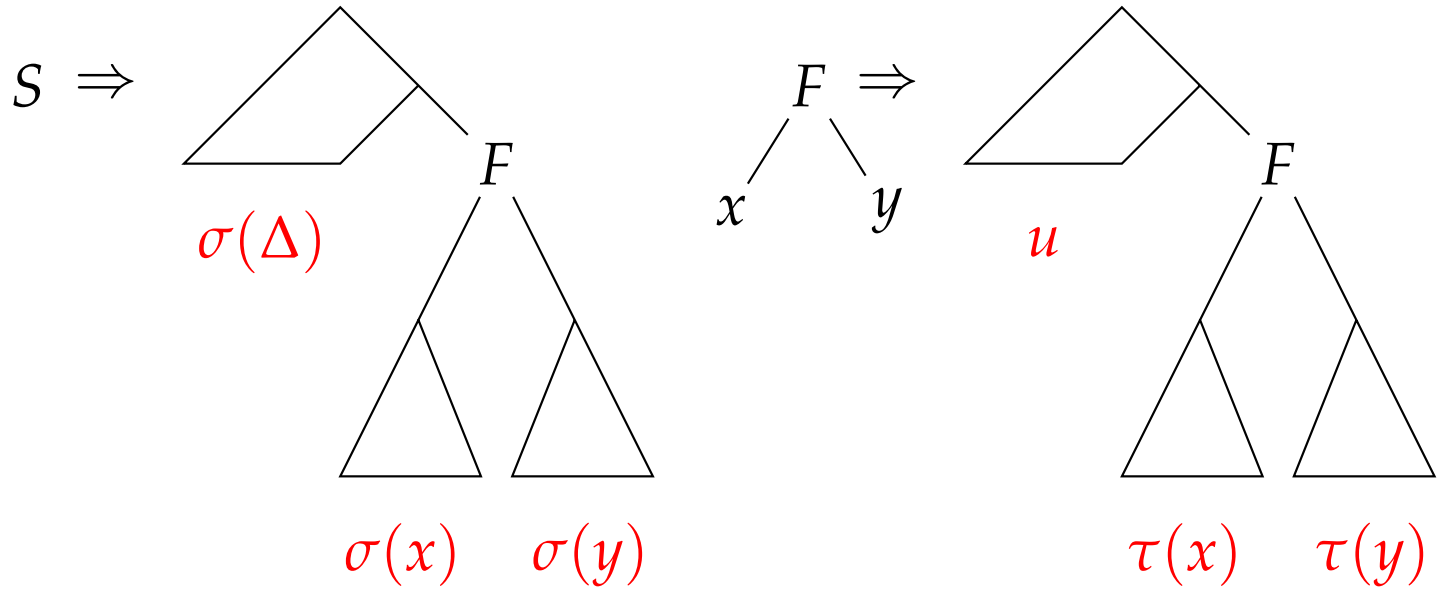

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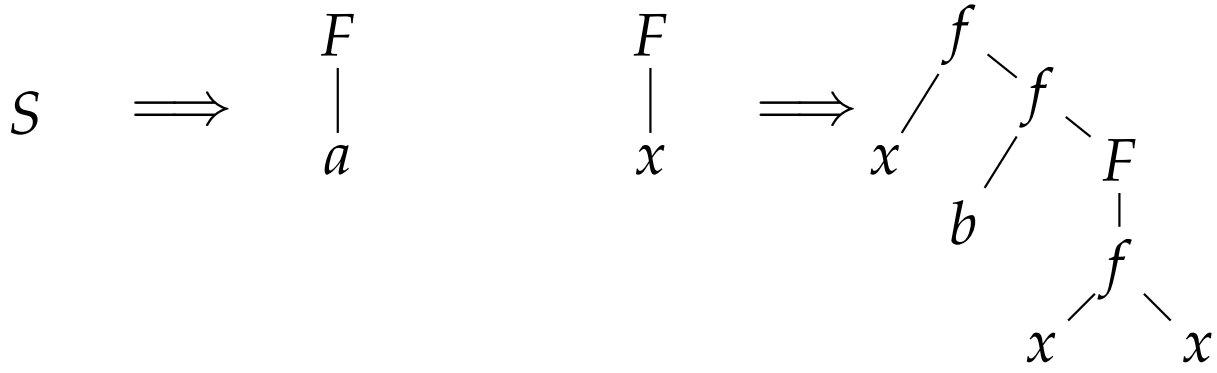

Letters : $\{\Delta, a, b, x\}$

$$\begin{aligned} \sigma(\Delta) &= \varepsilon & \qquad \tau(\Delta) &= \Delta x b \\ \sigma(a) &= a & \qquad \tau(a) &= a \\ \sigma(b) &= b & \qquad \tau(b) &= b \\ \sigma(x) &= a & \qquad \tau(x) &= xx \end{aligned}$$

Towards next-order morphic words

Can we expect to

- increase subword complexity?

Theorem (Allouche-Shallit)

The number of words of length n in a morphic word is at most $\mathcal{O}(n^2)$.

- increase growth rate?

Theorem (Carton-Thomas)

The sequence of indexes of a given letter in a morphic word is at most $\mathcal{O}(k^n)$ for some k.

Recursion schemes : next order

Instead of simply arity, symbols have fixed type (starting with a *base type* **o** :

- terminals $T = \{a, b, f, g \dots\}$,
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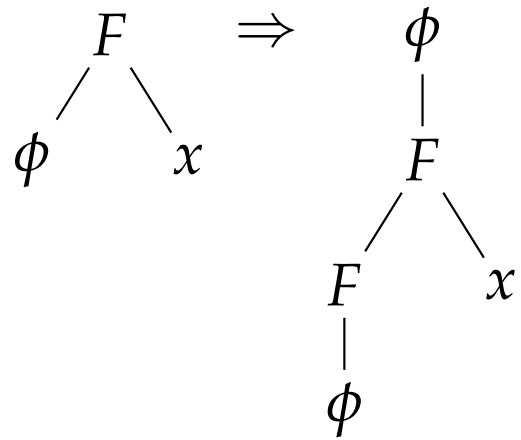

$$x \;:\; \mathbf{o} \qquad\qquad F \;:\; (\mathbf{o} \to \mathbf{o}) \to \mathbf{o} \to \mathbf{o}$$
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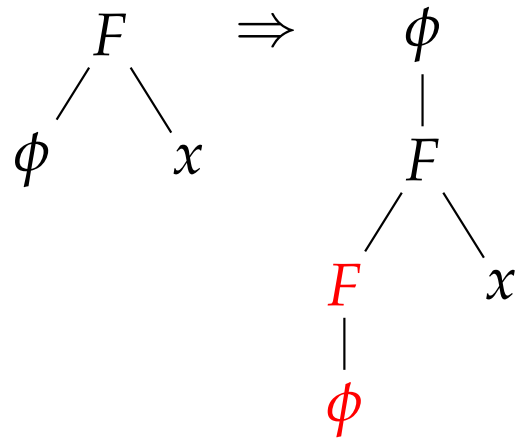


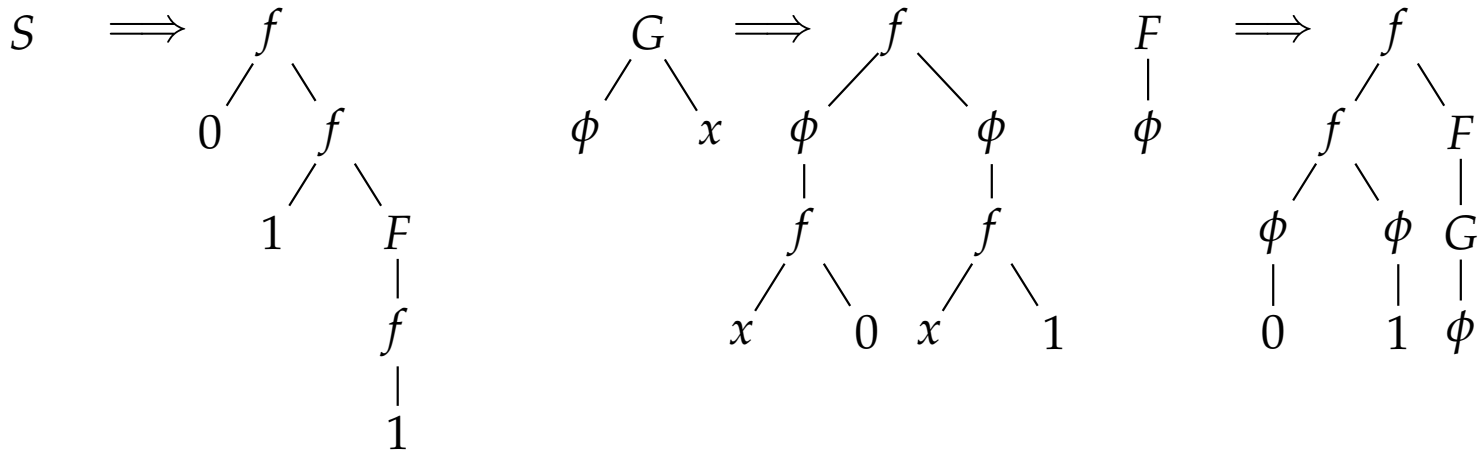
$$x : \mathbf{o} \qquad F : (\mathbf{o} \to \mathbf{o}) \to \mathbf{o} \to \mathbf{o}$$
$$\phi : \mathbf{o} \to \mathbf{o} \qquad F\phi : \mathbf{o} \to \mathbf{o}$$

Champernowne

The *Champernowne's constant* is simply the concatenation of numbers.

$$0123456778910 11\ldots$$
$$0110111001011 10\ldots$$

Champernowne : scheme approach



$$
\begin{array}{lcl}
S & : & \mathbf{o} \\
F & : & (\mathbf{o} \to \mathbf{o}) \to \mathbf{o} \\
G & : & (\mathbf{o} \to \mathbf{o}) \to \mathbf{o} \to \mathbf{o}
\end{array}
\qquad
\begin{array}{lcl}
f & : & \mathbf{o} \to \mathbf{o} \to \mathbf{o} \\
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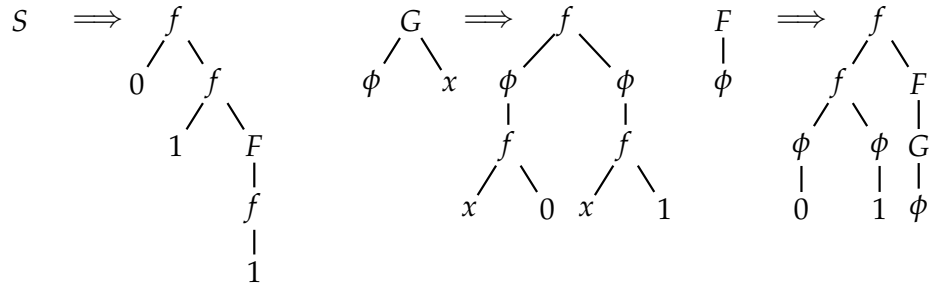

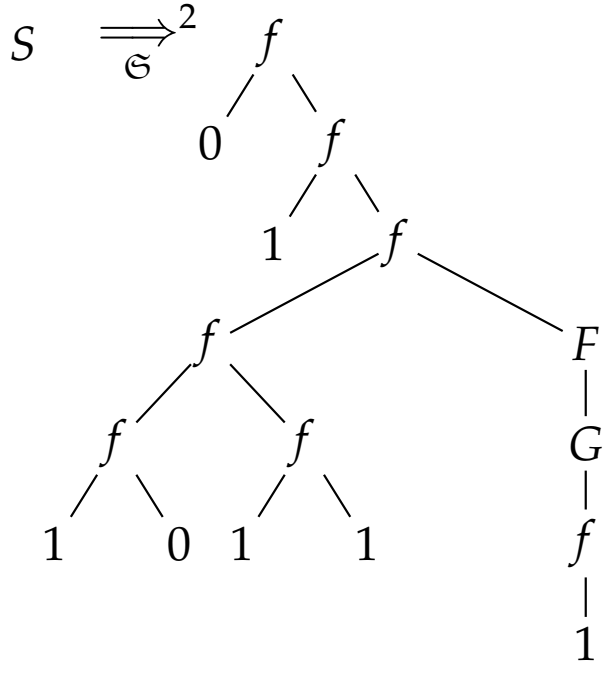

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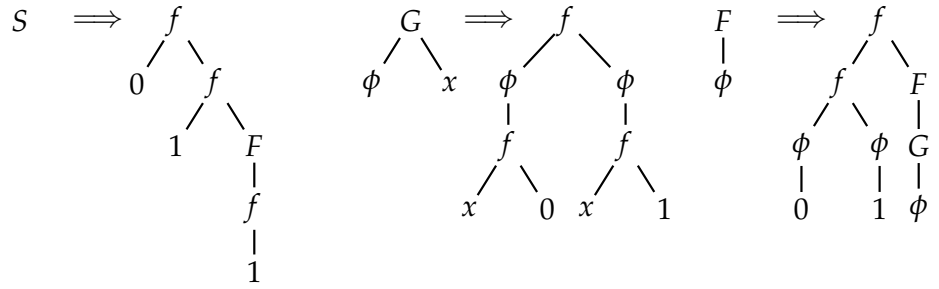

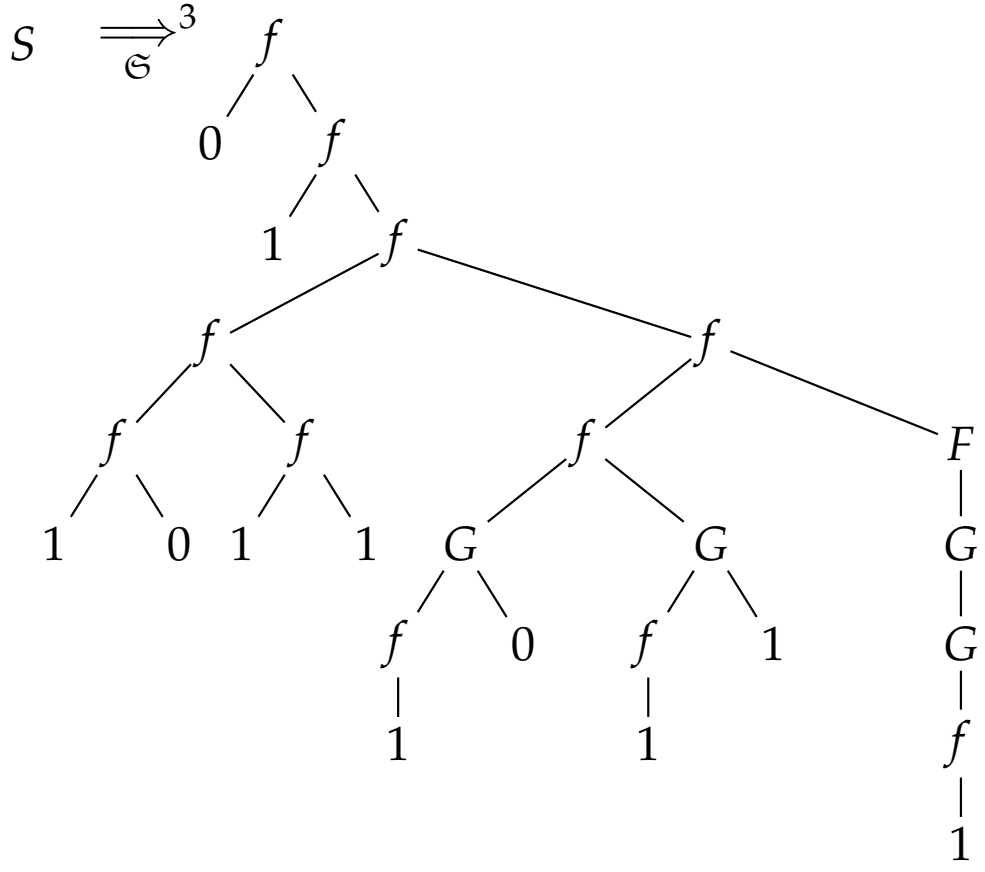

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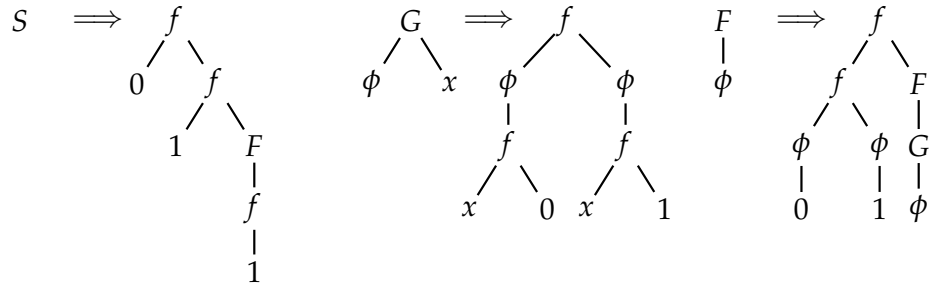

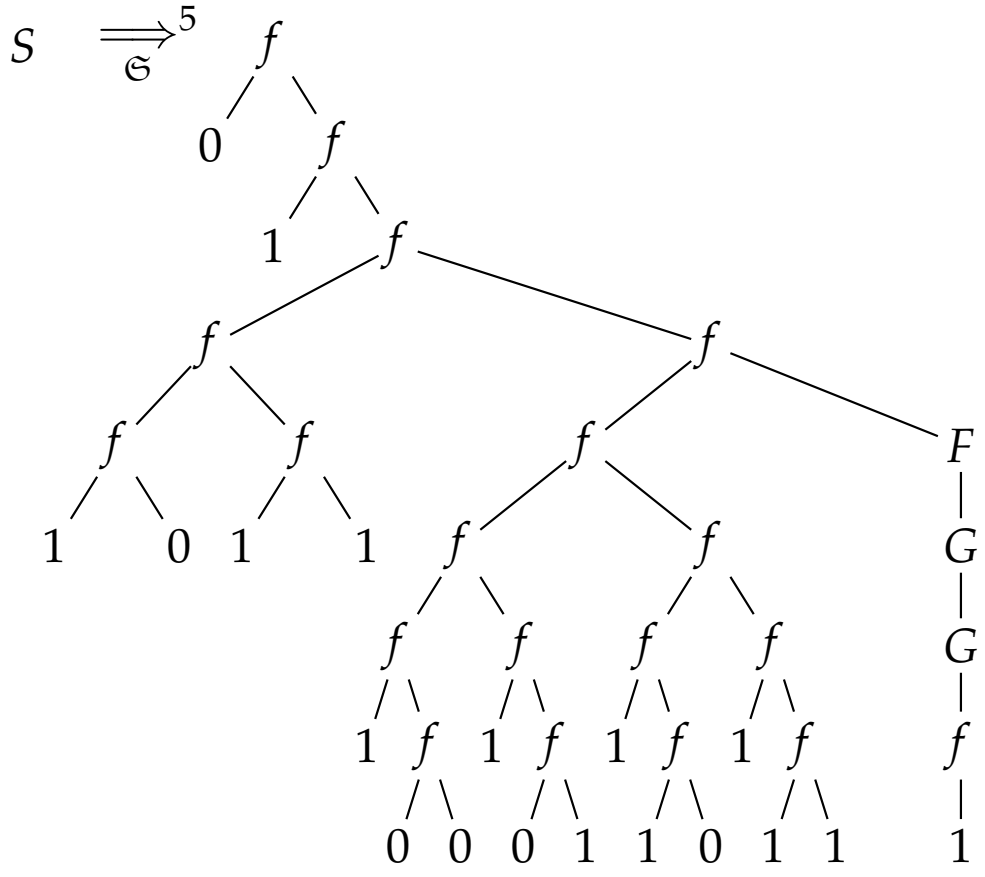

Linearization of order-2 schemes

Can we have the same linearization lemma as before?

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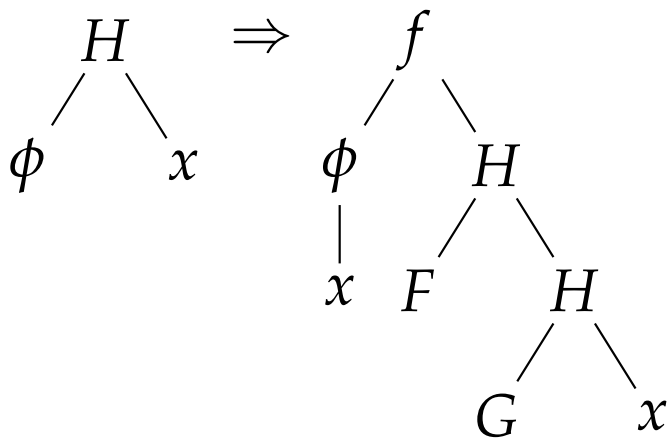
Lemma

For any scheme in $\mathcal{S}_2$, there is a scheme in $\mathcal{S}_2$ with only useful nonterminals generating the same tree.



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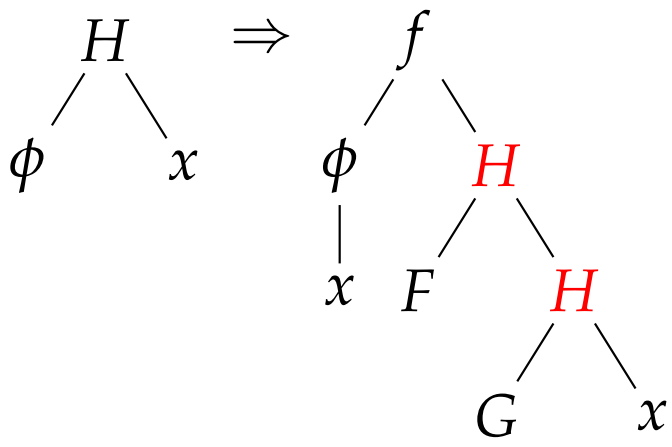

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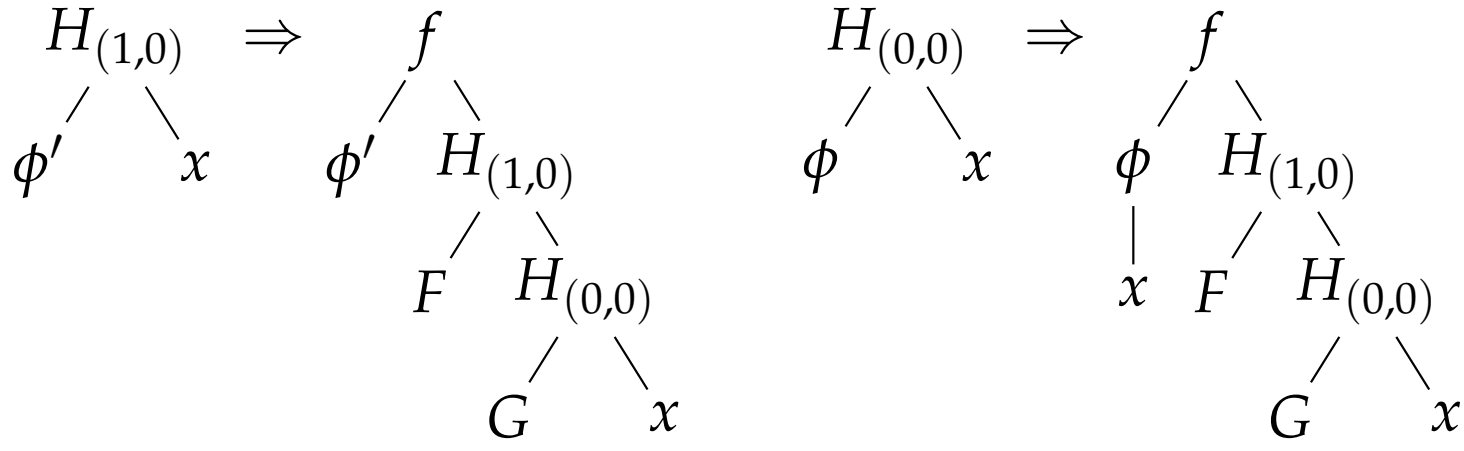


($H_{\overrightarrow{a}}$ means "argument i has arity recuded by a_i)

Linearization

Nonterminals are separated into

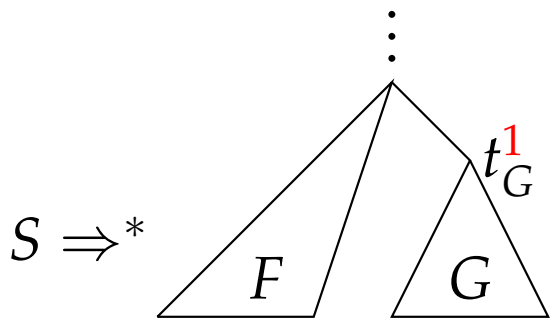
- *semiterminals* : nonterminals rewriting into finite trees,
- ∞-*nonterminals* the other ones.

Lemma (linearization)

For any order-2 scheme $\mathfrak{S}$ *generating a comb, there is* $\mathfrak{S}'$ *with only two* ∞*-nonterminals* $\{S, R\}$ *such that* $\mathbf{Fr}(\mathfrak{S}) = \mathbf{Fr}(\mathfrak{S}')$. *Moreover, their rewriting rules have exactly one occurence of* R *and none of* S.

Linearization

Given the shape of the tree, we actually never have two ∞-nonterminals at the same time.



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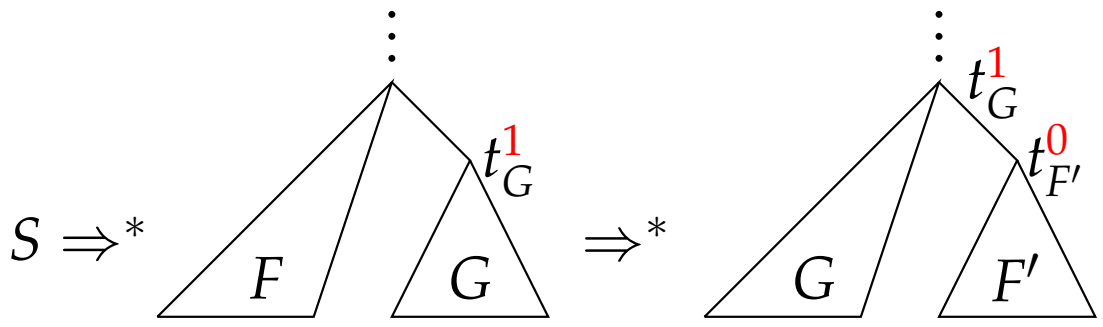
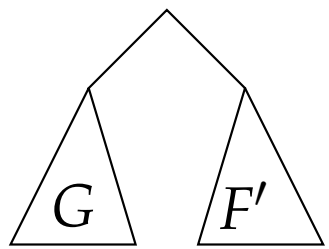
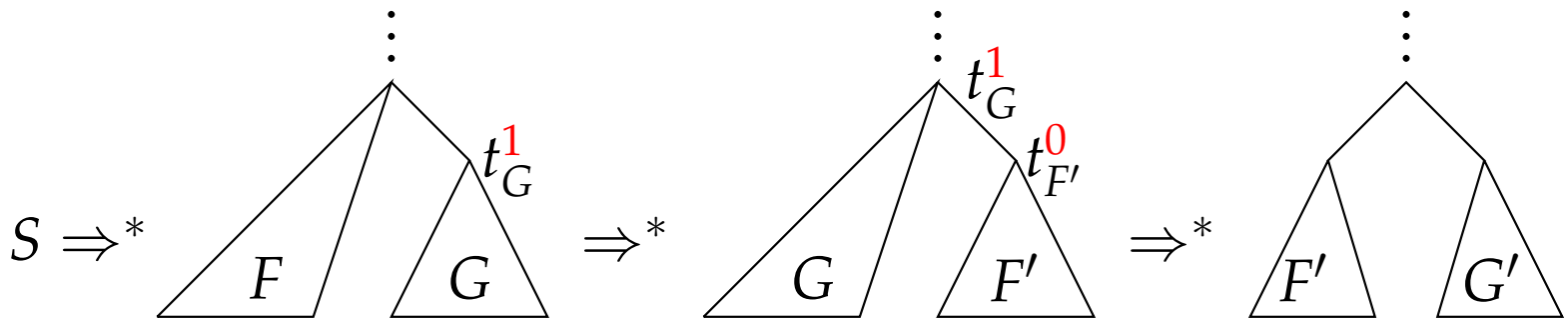


or

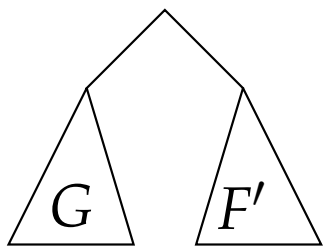


Linearization

Given the shape of the tree, we actually never have two ∞-nonterminals at the same time.



or



Term words

Alphabet $\Sigma = \bigcup_{i=0}^{n} \Sigma_i$ where Σ_0 is called "letters".

$$\theta := \epsilon \mid a \in \Sigma_0 \mid f(\underbrace{\theta, \ldots, \theta}_{i}), f \in \Sigma_i \mid \theta \cdot \theta$$

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$\Sigma(\bar{z}) = \{f(z_1, \ldots, z_k) \mid f \in \Sigma_k\}$.
Let τ, σ be two morphisms on $\Sigma(\bar{z})^*$ w.r.t. concatenation.

for $f \in \Sigma_k$ and $z_1, \ldots, z_k \in \mathcal{V}$,

$$\begin{aligned} \tau(f(z_1, \ldots, z_k)) &\in \mathsf{TW}(\Sigma \cup \{z_1, \ldots, z_k\}) \\ \sigma(f(z_1, \ldots, z_k)) &\in \mathsf{TW}(\Sigma_0 \cup \{z_1, \ldots, z_k\}) \\ &= (\Sigma_0 \cup \{z_1, \ldots, z_k\})^* \end{aligned}$$

2-morphic words

This definition is extended on term words by

$$\begin{aligned} &\text{for } f \in \Sigma_k \\ &\text{and } t_1, \ldots, t_k \in \mathsf{TW}(\Sigma), \\ \tau(f(t_1, \ldots, t_k)) &= \tau(f(z_1, \ldots, z_k))[\forall i, z_i := \tau(t_i)] \\ \sigma(f(t_1, \ldots, t_k)) &= \sigma(f(z_1, \ldots, z_k))[\forall i, z_i := \sigma(t_i)] \end{aligned}$$

Let $\Delta \in \Sigma_0$, words of the form $\sigma(\tau^\omega(\Delta))$ are *2-morphic words*.

Champernowne : 2-morphic words

$\Sigma_0 = \{0, 1\}$ and $\Sigma_1 = \{g\}$.

$$\begin{aligned} \tau(\Delta) &= \Delta g(0) g(1) \\ \tau(g(z)) &= g(z0) g(z1) \\ \sigma(\Delta) &= 01 \\ \sigma(g(z)) &= 1z \end{aligned}$$

In addition $\tau(1) = \sigma(1) = 1$ and $\tau(0) = \sigma(0) = 0$.

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$$\begin{aligned} \tau(\Delta) &= \Delta \quad g(0) \quad g(1) \\ \tau^{(2)}(\Delta) &= \Delta \quad g(0) \quad g(1) \quad g(00) \quad g(01) \quad g(10) \quad g(11) \end{aligned}$$

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$$\begin{array}{rcccccccc} \tau(\Delta) & = & \Delta & g(0) & g(1) & & & & \\ \tau^{(2)}(\Delta) & = & \Delta & g(0) & g(1) & g(00) & g(01) & g(10) & g(11) \\ \sigma(\tau^{(2)}(\Delta)) & = & 01 & 10 & 11 & 100 & 101 & 110 & 111 \end{array}$$

Final result

Theorem

The frontiers of combs generated by order-2 schemes are exactly 2-morphic words.

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A *safe* scheme : in every rule $F\,\overrightarrow{x} \Rightarrow T_F$, and every subterm t of T_F, the order of t is lower of equal to any order of x_i inside it.
The proof of the theorem translates words into safe schemes.

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The proof of the theorem translates words into safe schemes.

Moreover, by MSO properties of the pushdown hierarchy, 2-morphic words are also

- ω-frontiers of safe trees,
- paths generated by order-3 safe schemes.

What about unsafe ones?

Consequences

We apply properties of the pushdown hierarchy.

Corollary

For any 2-morphic word w,

- *the MSO theory of w is decidable;*
- *for any MSO-transduction $\mathcal{T}$, if $\mathcal{T}(w)$ is a word, it is a 2-morphic word.*
- *the sequence of indexes of a given letter in a morphic word is at most $\mathcal{O}(2^{2^{Cn}})$ for some C. The bound is tight.*

Other example : characteristic word of $(n!)_{n\geq 0}$, known as the *Liouville constant*.

Conclusion and beyond

This construction builds a new class of graphs for order-2 schemes.

- What about higher orders? can we still linearize ?
- Connexion with classes $\mathbb{S}_k$ of [Fratani-Senizergues], or k-automatic words by [Bárány] ?
- And beyond the pushdown/Caucal/scheme hierarchy? the characteristic word of

$$\left(2^{2^{2^{\cdots}}\}n}\right)_{n\geq 0}$$

has decidable MSO-theory [Thomas]