The domino problem for fractal subsets between $\mathbb{Z}$ and $\mathbb{Z}^2$

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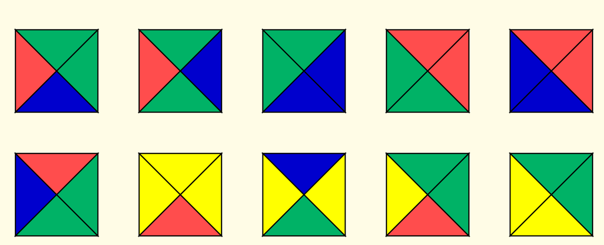
Classical Background

The domino problem was introduced by Wang [3] in 1961. It consists of deciding if copies of a finite set of Wang's tiles (square tiles of equal size, not subject to rotation and with colored edges) can tile the plane subject to the condition that two adjacent tiles possess the same color in the edge they share.

Example: Can these Wang tiles cover the plane?



Let $\mathcal{A}$ be a finite alphabet and $\mathcal{F}$ a set of patterns (functions from a finite part of $\mathbb{Z}^2$ to $\mathcal{A}$). The subshift $X_{\mathcal{F}}$ consists of all configurations $x : \mathbb{Z}^2 \to \mathcal{A}$ such that no translation of a pattern from $\mathcal{F}$ appears in them. In this context, the domino problem is defined as follows:

$$\mathtt{DP} = \{\langle \mathcal{F} \rangle \mid \mathcal{F} \text{ is a finite set of patterns}, X_{\mathcal{F}} \neq \emptyset\}.$$

Theorem [Berger, 66']

$\mathtt{DP}$ is undecidable.

These notions are easily generalized when $\mathbb{Z}^2$ is replaced by a finitely generated group. We denote the domino problem for a group G as $\mathtt{DP}(G)$.

Some facts about $\mathtt{DP}(G)$.

- $\mathtt{DP}(\mathbb{Z}^d)$ is decidable $\iff$ $d = 1$.
- $\mathtt{DP}(G)$ is decidable for virtually free groups
- $\mathtt{DP}(G)$ is undecidable if G has undecidable word problem.
- Baumslag solitar groups have undecidable domino problem [Aubrun, Kari 2013]
- $\mathtt{DP}(G)$ is characterized for polycyclics groups [Jeandel, 2015]
- $\mathtt{DP}(G)$ is a quasiisometry invariant for finitely presented groups [Cohen, 2015]

Abstract

We define the domino problem for tilings over self-similar structures of $\mathbb{Z}^2$ given by forbidden patterns. In this setting we exhibit non-trivial families of subsets with decidable and undecidable domino problem.

Problem setting

Ingredients

- s: a $\mathbb{Z}^2$-substitution over the alphabet $\{\square, \blacksquare\}$.
- $\mathcal{A}$ a finite alphabet.
- $\mathcal{F}$ a finite set of forbidden patterns over $\mathcal{A}$.

Recipe

We consider X_s the subshift generated by the substitution s: The set of configurations $x \in \{\square, \blacksquare\}^{\mathbb{Z}^2}$ such that all patterns in x appear in some iteration of s.

We define $X_s(\mathcal{F})$ as the set of colorings by $\mathcal{A}$ of the black boxes of configurations in X_s such that no patterns in $\mathcal{F}$ appear. These can be identified as configurations over $(\mathcal{A} \cup \{0\})^{\mathbb{Z}^2}$ which project onto X_s.

When s sends the white block to an array of white blocks, the resulting structure is self-similar (we call it a self-similar substitution). Thus $X_s(\mathcal{F})$ is the set of colorings of a self-similar structure. We can define its domino problem as follows:

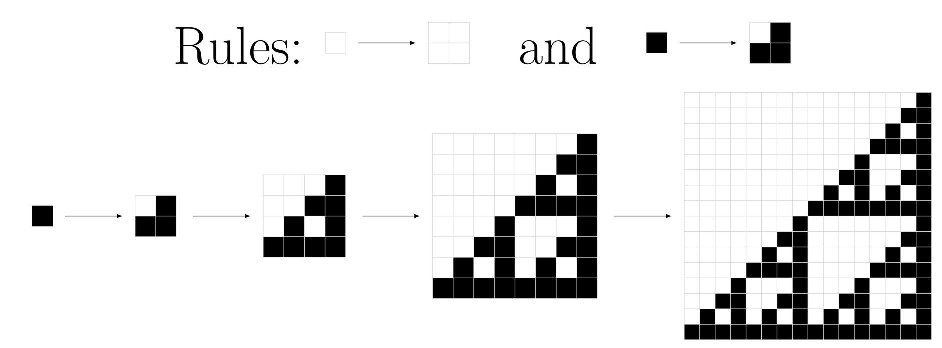
$$\mathtt{DP}(s) := \{\langle \mathcal{F} \rangle \mid \mathcal{F} \text{ is finite } : X_s(\mathcal{F}) \neq \{0^{\mathbb{Z}^2}\}\}.$$

Question

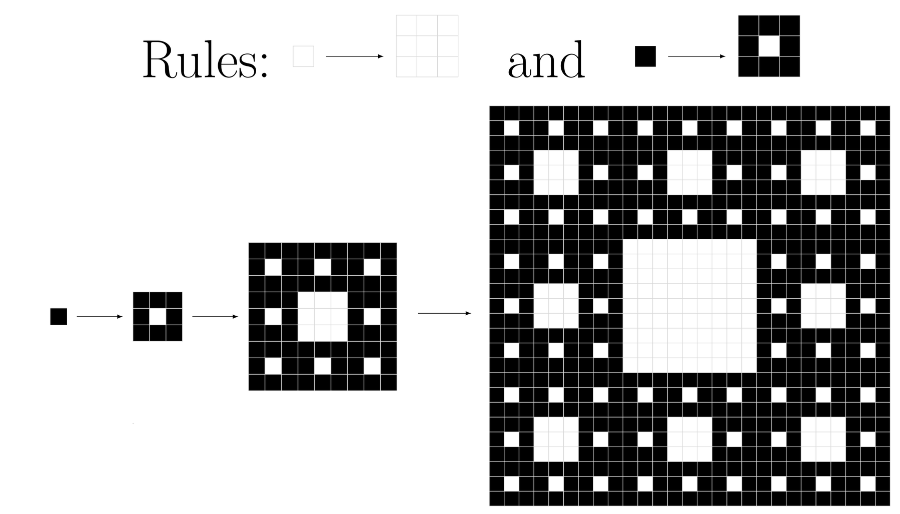
Can we characterize for which self-similar substitutions $\mathtt{DP}(s)$ is decidable ?

Some self-similar substitutions

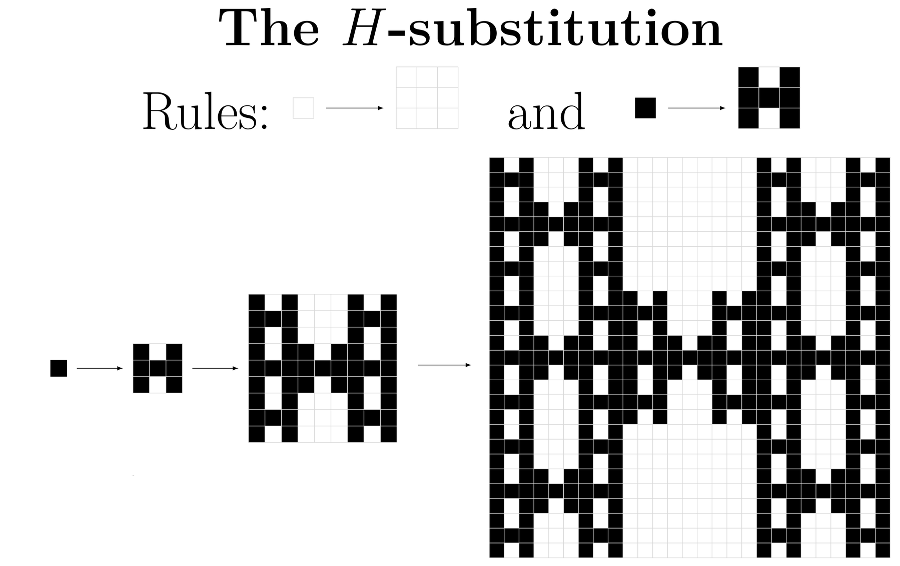
Sierpiński triangle substitution



Sierpiński carpet substitution

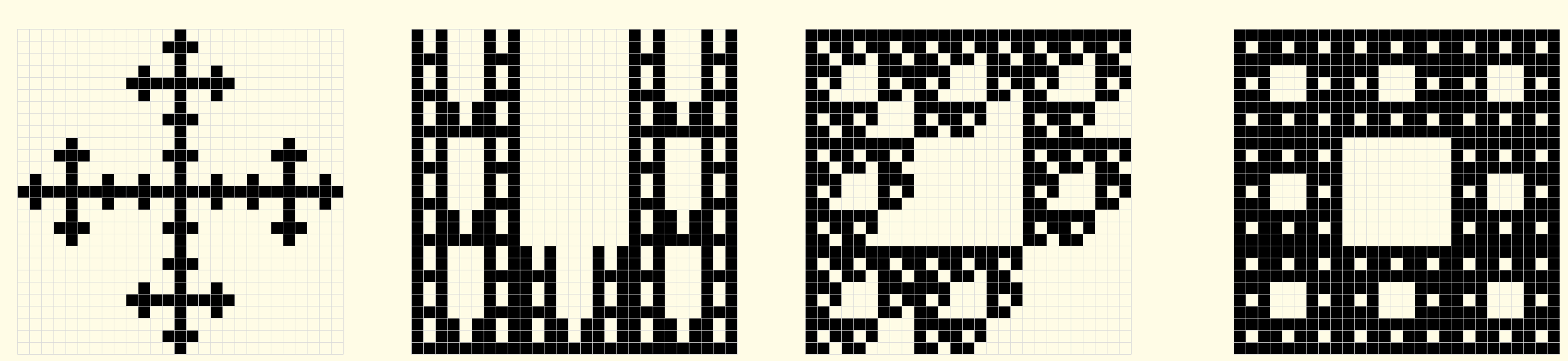


The H-substitution



Four connectivity classes

B. Connectivity | **Isthmus** | **Weak grid** | **Strong grid**



Some answers

First case: Bounded connectivity.

The domino problem is decidable for bounded connectivity substitutions.

► This includes the Sierpiński triangle.

Second case: Grids.

The domino problem is undecidable for weak and strong grid substitutions.

► This includes the Sierpiński carpet.

Note: The proof is much harder for weak grids.

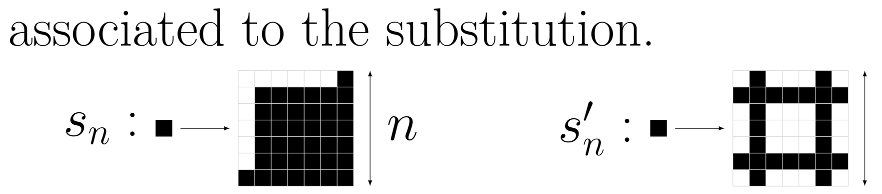
Third case: Isthmus.

We don't know whether the domino problem is undecidable or not for Isthmus substitutions!

► This includes the H-substitution.

Further remarks

The decidability of the domino problem is not related at all with the Hausdorf dimension of the fractal associated to the substitution.



Some of the previous proofs rely on a natural version of Mozes's theorem [2] in this setting: Namely, that the systems generated by substitutions can be defined using local rules. This property is satisfied by all grids and some bounded connectivity substitutions. We don't know if it's satisfied by isthmuses.

References

[1] Berger, R. *The Undecidability of the Domino Problem* American Mathematical Society, 1966.

[2] Mozes, S. *Tilings, substitution systems and dynamical systems generated by them.* Journal d'Analyse Mathématique, 1989.

[3] Wang, H. *Proving Theorems by Pattern Recognition II* Commun. ACM, 1961.