The domino problem for structures between $\mathbb{Z}$ and $\mathbb{Z}^2$.

Sebastián Barbieri
LIP, ENS de Lyon

Paris-Est-Créteil
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G-subshifts

Consider a group G.

- $\mathcal{A}$ is a finite alphabet. Ex : $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of functions $x : G \to \mathcal{A}$.
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the shift action given by :

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Alternative definition : G-subshift

X is a G-subshift if it can be defined as the set of configurations which avoid a set forbidden patterns : $\exists \mathcal{F} \subset \bigcup_{F \subset G, |F| < \infty} \mathcal{A}^F$ such that :

$$X = X_{\mathcal{F}} := \{x \in \mathcal{A}^G \mid \forall p \in \mathcal{F} : p \not\sqsubset x\}.$$

Example in $\mathbb{Z}^2$: Fibonacci shift

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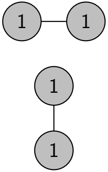
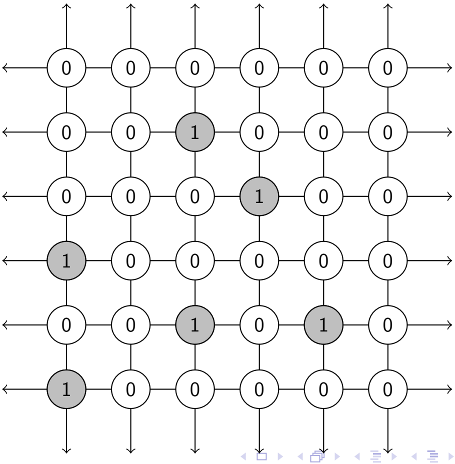
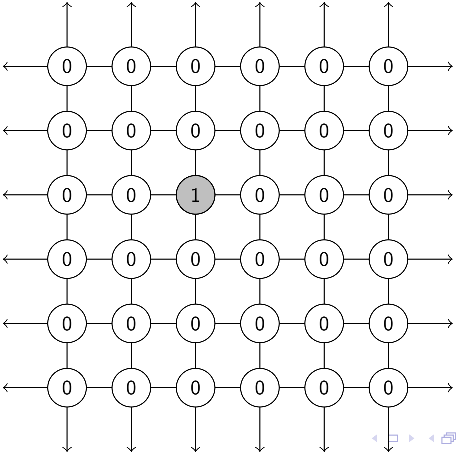


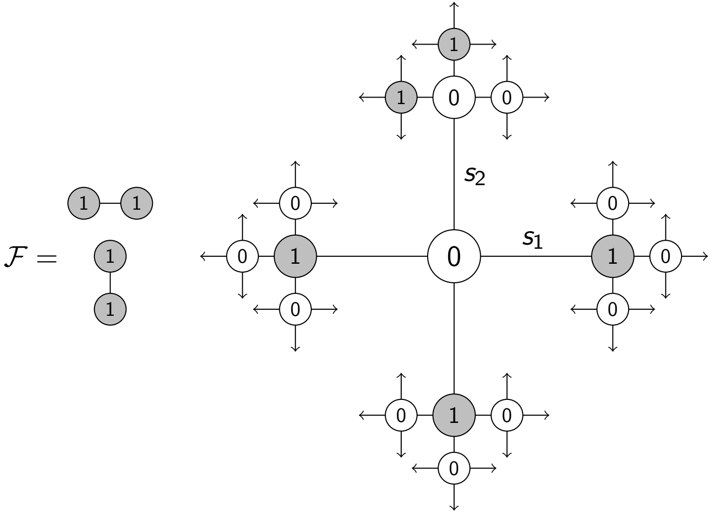


Example : one-or-less subshift.

$$X_{\leq 1} := \{x \in \{0,1\}^{\mathbb{Z}^d} \mid |\{z \in \mathbb{Z}^d : x_z = 1\}| \leq 1\}.$$



Fibonacci in F_2.



Subshifts of finite type.

- What about if we only consider local rules ?

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A G-subshift is of finite type (SFT) if it can be defined by a finite set $\mathcal{F}$ of forbidden patterns.

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- Given a finite set of forbidden patterns, can we decide if the G-subshift produced by them is non-empty ?

The domino problem.

- Every finite alphabet can be identified as a finite subset of $\mathbb{N}$.

Domino problem.

$$\mathtt{DP}(G) = \{\mathcal{F} \subset \mathbb{N}_G^* \mid |\mathcal{F}| < \infty, X_{\mathcal{F}} \neq \emptyset\}.$$

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- if G is finitely generated by the set S, we can codify each pattern as a function from a finite set of words in $(S \cup S^{-1})^*$ to $\mathbb{N}$.
- Therefore, $\mathtt{DP}(G)$ can be written as a formal language. We say G has decidable domino problem if $\mathtt{DP}(G)$ is Turing-decidable.

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Question : Which groups have decidable domino problem ?

The easy case $G = \mathbb{Z}$.

Theorem :

The set of configurations of a $\mathbb{Z}$-SFT can be characterized as the set of bi-infinite walks in a finite graph.

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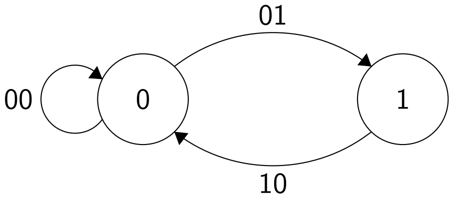

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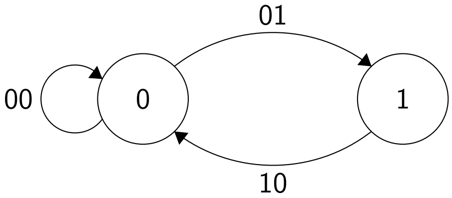

As the graph is finite, a $\mathbb{Z}$-SFT is non-empty if and only if its Rauzy graph contains a cycle, thus $\mathtt{DP}(\mathbb{Z})$ is decidable.

The not so easy case : $G = \mathbb{Z}^2$

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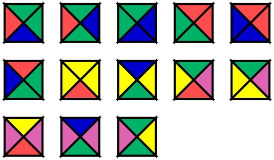
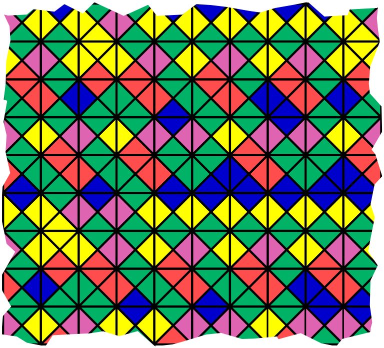
The name "Domino problem" comes from the $G = \mathbb{Z}^2$ case. Wang tiles are unit squares with colored edges, the forbidden patterns are implicit in the alphabet.





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If Wang's conjecture is true, we can decide if a set of Wang tiles can tile the plane !

Semi-algorithm 1 :

1. Accept if there is a periodic configuration.
2. loops otherwise

Semi-algorithm 2 :

1. Accept if a block $[0, n]^2$ cannot be tiled without breaking local rules.
2. loops otherwise

Wang's conjecture

Theorem[Berger 1966]

Wang's conjecture is FALSE

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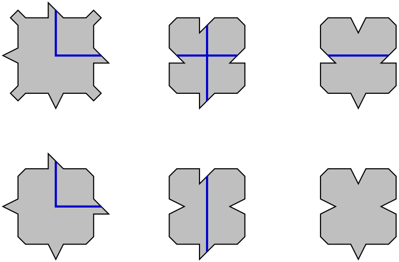
His construction encodes a Turing machine using an alphabet of size 20426.

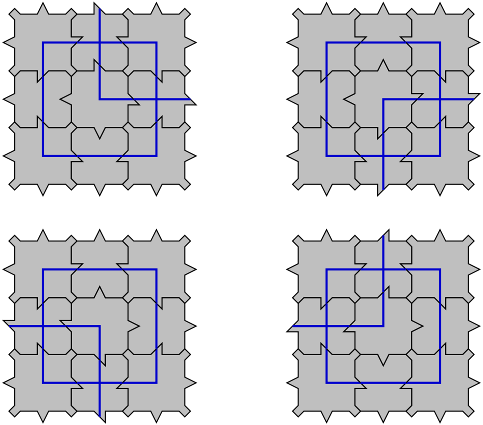
His proof was later simplified by Robinson[1971]. A proof with a different approach was also presented by Kari[1996].

Robinson tileset

The Robinson tileset, where tiles can be rotated.

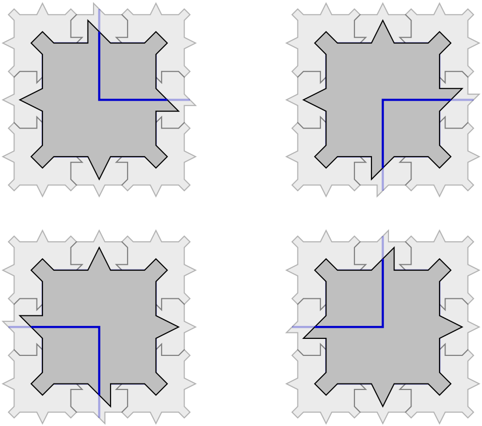


Macro-tiles of level 1.



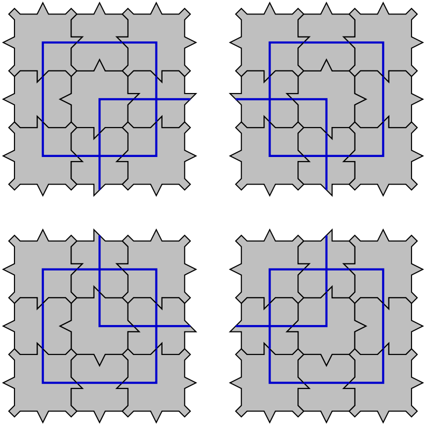
General structure of the Robinson tiling

Macro-tiles of level 1.



They behave like large .

From macro-tiles of level 1 to macro-tiles of level 2



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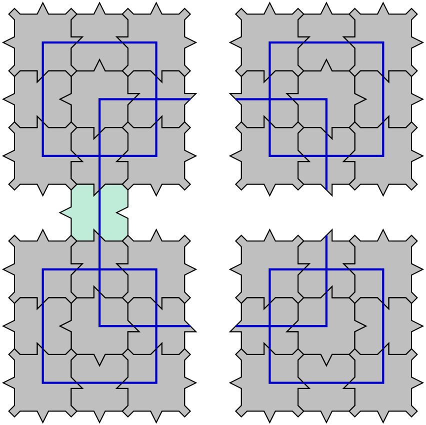

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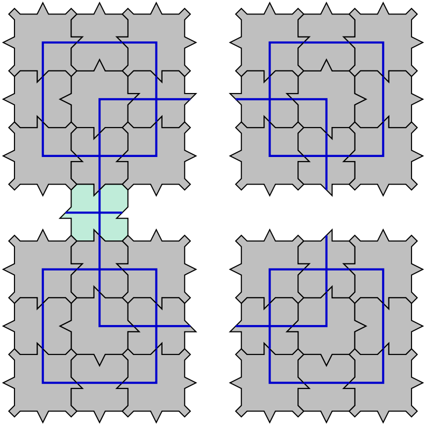

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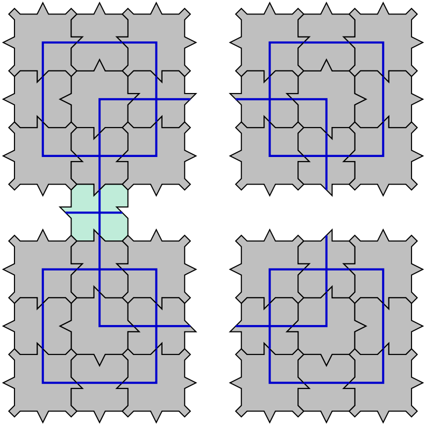

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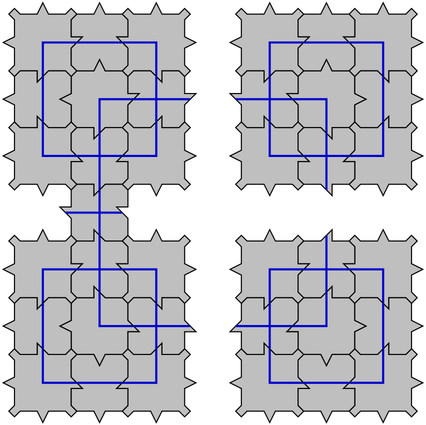

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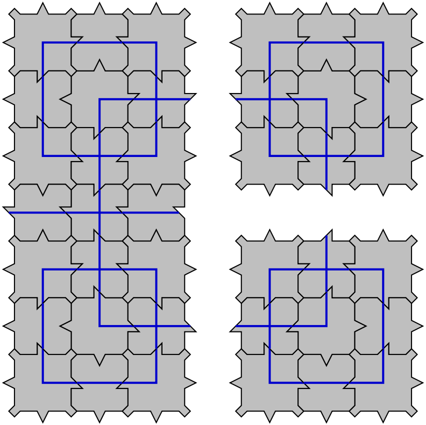

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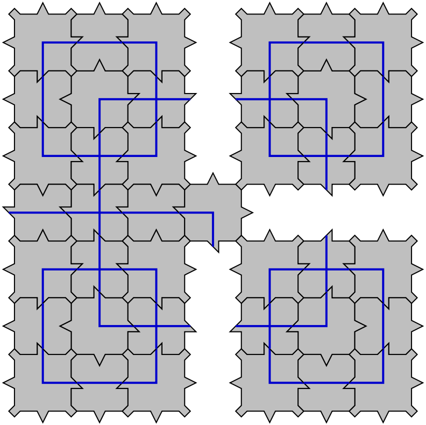

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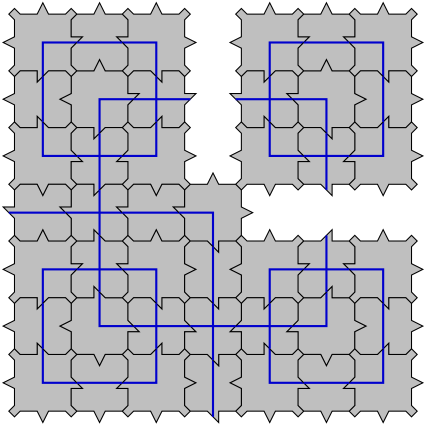

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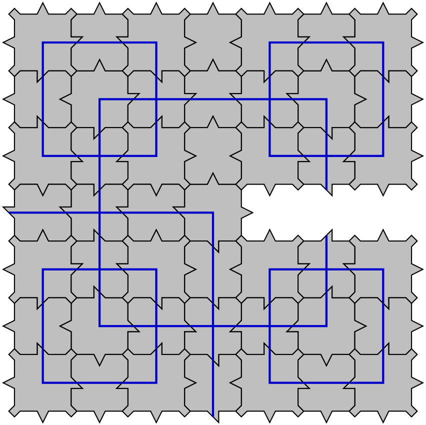

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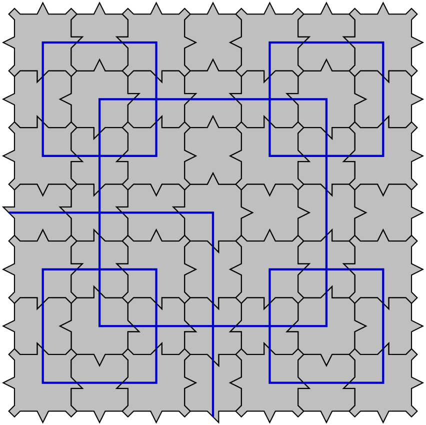

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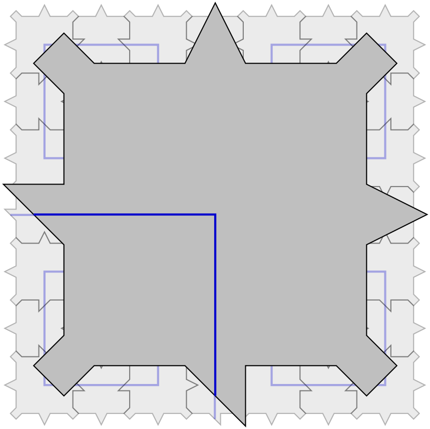
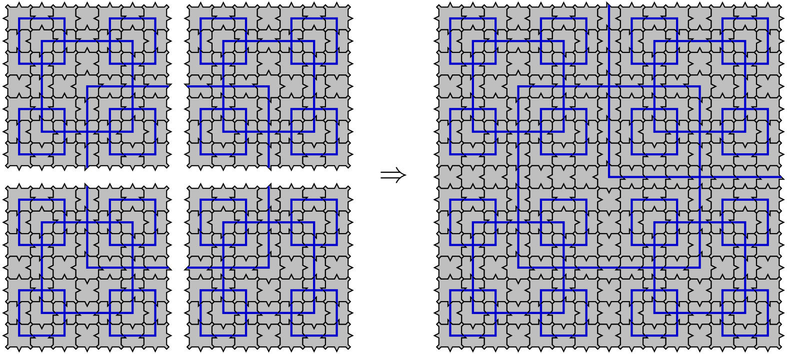


From macro-tiles of level n to macro-tiles of level $n + 1$



Some recent results and facts in f.g. groups

- If a group G has undecidable word problem $\Rightarrow \mathtt{DP}(G)$ is undecidable.
- Virtually free groups have decidable domino problem.
- For virtually nilpotent groups : $\mathtt{DP}(G)$ is decidable if and only if it has two or more ends (2013 Ballier, Stein).
- Every virtually polycyclic group which is not virtually $\mathbb{Z}$ has undecidable domino problem (work in progress by Jeandel).
- The domino problem is a quasi-isometry invariant for finitely presented groups (2015 Cohen).

Going between $\mathbb{Z}$ and $\mathbb{Z}^2$ (Joint work with M. Sablik)

So far we have :

- $\mathtt{DP}(\mathbb{Z})$ is decidable.
- $\mathtt{DP}(\mathbb{Z}^2)$ is undecidable.

And if $H \leq \mathbb{Z}^2$, then either $H \cong 1, H \cong \mathbb{Z}$ or $H \cong \mathbb{Z}^2$.

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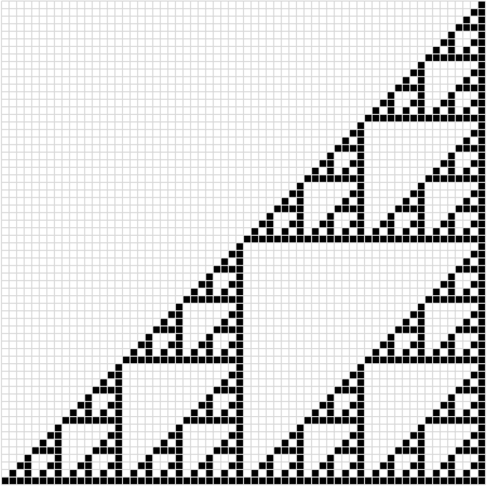
We need to lose the group structure if we want to study intermediate structures.

Toy case : Sierpiński triangle



Coding subsets of $\mathbb{Z}^2$ as configurations.

Let $F \subset \mathbb{Z}^2$ and define the configuration $x_F \in \{0,1\}^{\mathbb{Z}^2}$:

$$(x_F)_z = \begin{cases} 1 \text{ if } z \in F \\ 0 \text{ if not.} \end{cases}$$

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And let $Y = \overline{\bigcup_{z \in \mathbb{Z}^2} \{\sigma_z(x_F)\}}$.
Given a set of forbidden patterns $\mathcal{F}$ we can define colorings of F as the configurations of $X_{\mathcal{F}}$ over an alphabet $\mathcal{A} \ni 0$ such that the application $\pi : \mathcal{A}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$:

$$\pi(x)_z = \begin{cases} 1 \text{ if } x_z \neq 0 \\ 0 \text{ if } x_z = 0 \end{cases}$$

yields an element of Y.

Formally...

- Let $Y \ni 0^{\mathbb{Z}^2}$ be a $\mathbb{Z}^2$-subshift over the alphabet $\{0,1\}$.
- Let $\mathcal{F}$ be a set of forbidden patterns over an alphabet $\mathcal{A} \ni 0$ which does not forbid any pattern consisting only of 0.

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Definition : Y-based domino problem

$$\mathtt{DP}(Y) := \{\mathcal{F} \subset \mathbb{N}^*_{\mathbb{Z}^2} \mid |\mathcal{F}| < \infty \text{ and } X_{Y,\mathcal{F}} \neq \{0^{\mathbb{Z}^2}\}\}.$$

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- If Y contains a strongly periodic point which is not $0^{\mathbb{Z}^2}$ then $\mathtt{DP}(Y)$ is undecidable.
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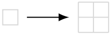
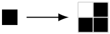
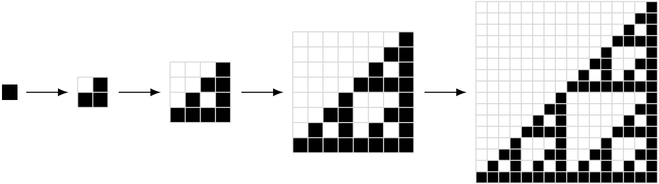
In particular we consider : substitutions over $\{0, 1\}$ such that the image of 0 is a rectangle of zeros.

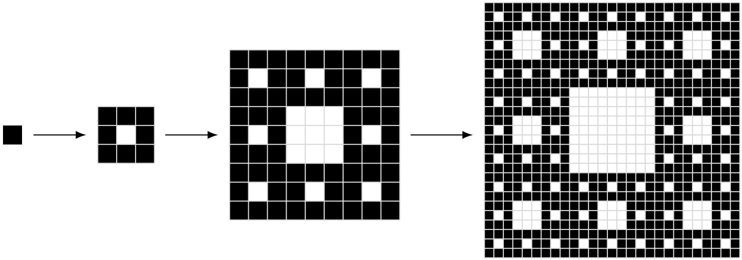
Example 1 : Sierpiński triangle

Consider the alphabet $\mathcal{A} = \{\square, \blacksquare\}$ and the self-similar substitution s such that :

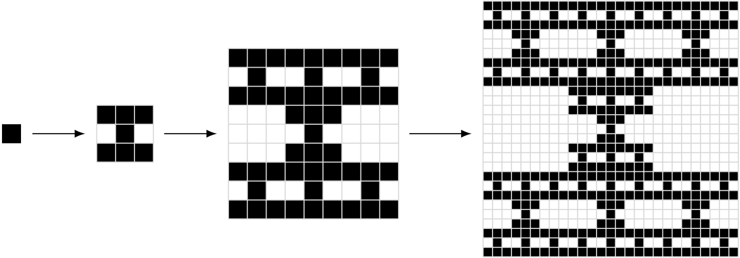
 and 



Example 2 : Sierpiński carpet



Example 3 : The Bridge.



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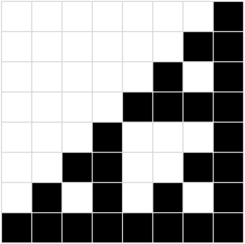

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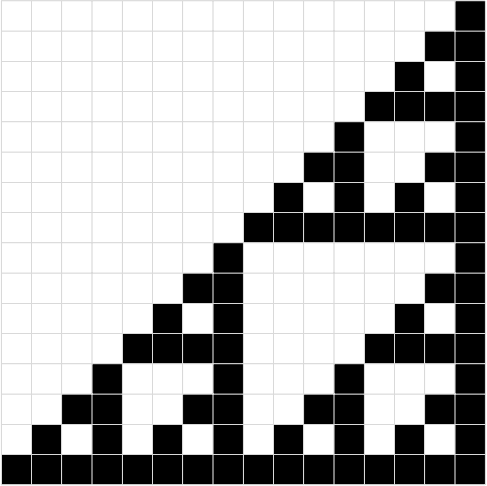

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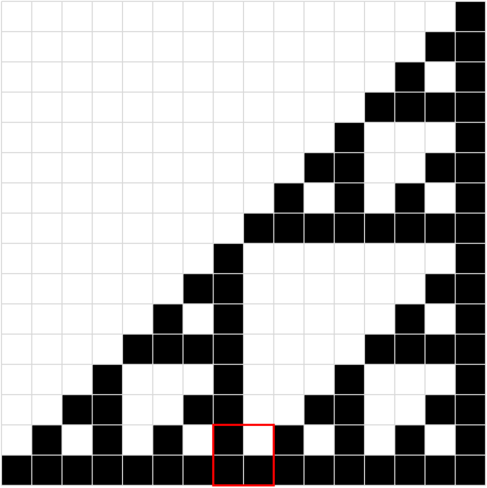

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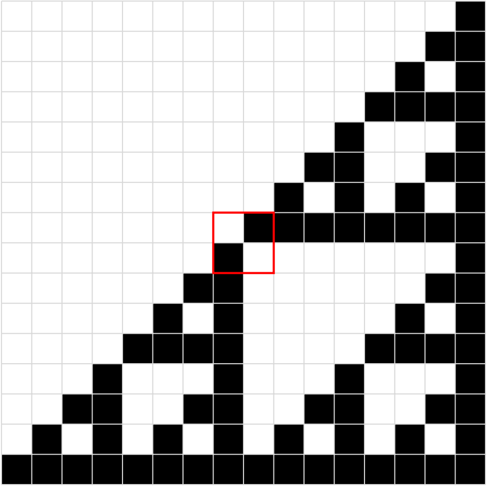

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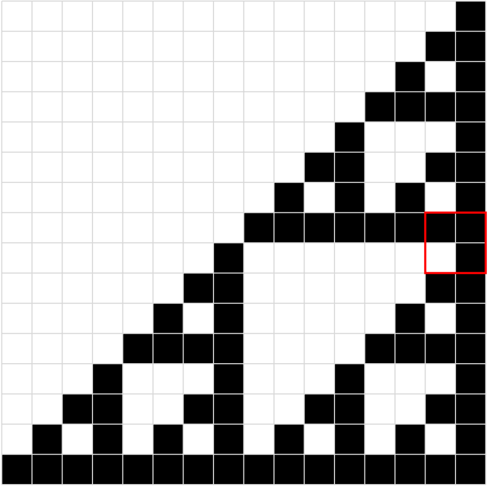

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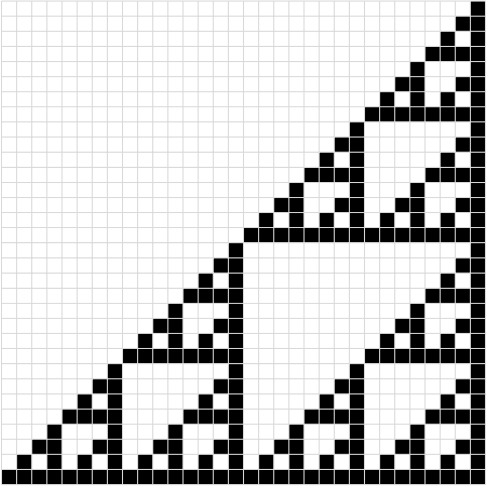

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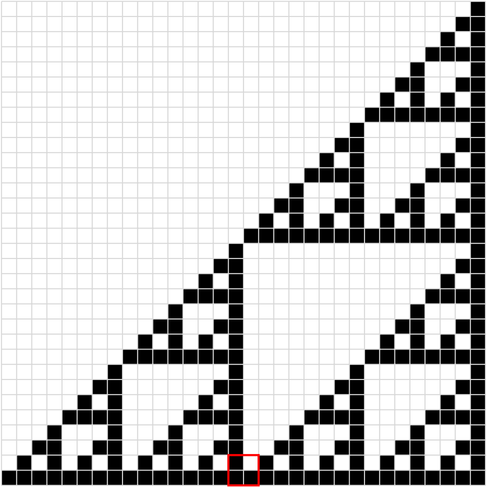

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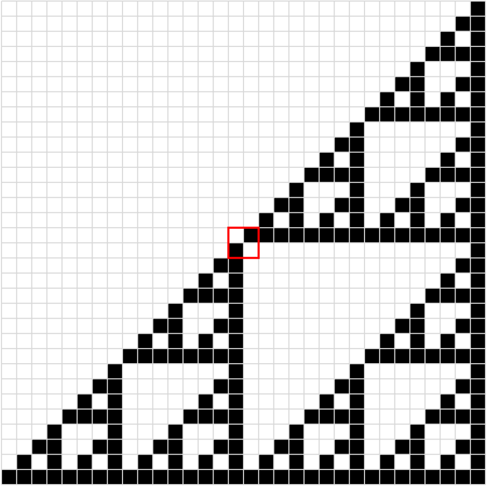

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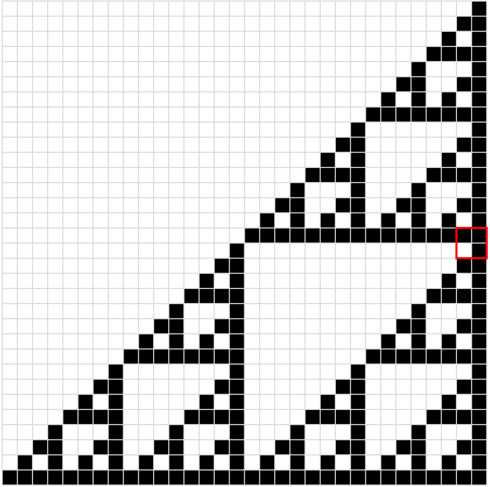

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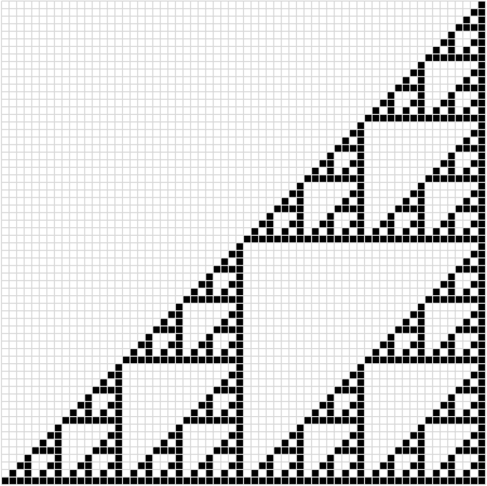

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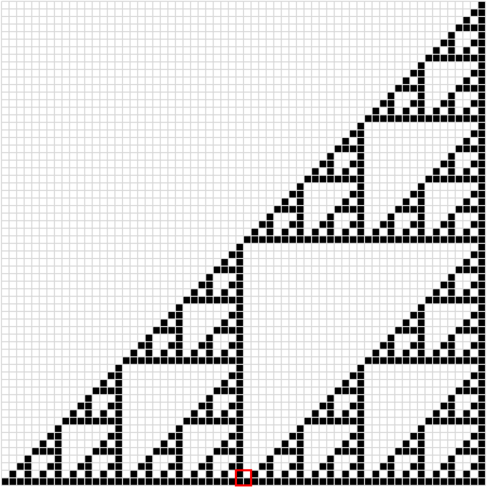

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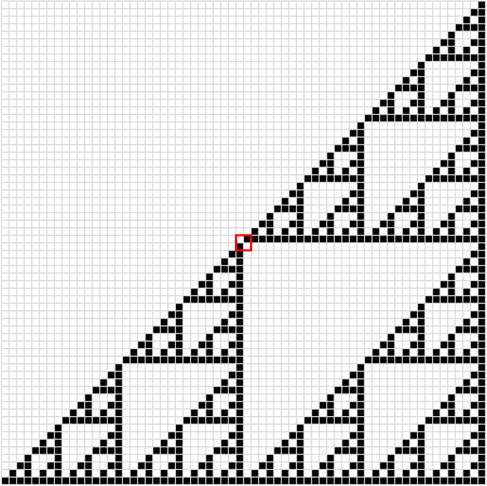

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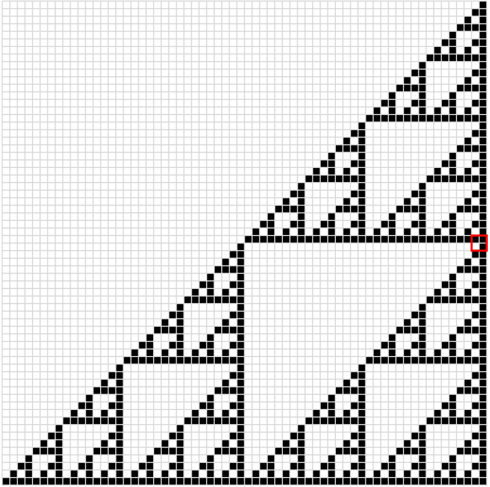

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Theorem :

The domino problem is decidable in the Sierpiński triangle.

Proof strategy (continued) :

- Keep the information about the pasting places (finite tuples) and build pasting rules $(T_1, T_2, T_3) \to T_4$.

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This technique can be extended to a big class of self-similar substitutions !

Toy case 2 : Sierpiński carpet.

Theorem :

The domino problem is undecidable in the Sierpiński carpet.

Proof strategy :

- Suppose we can simulate substitutions over the Sierpiński carpet (using a bigger alphabet).

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$$\bullet \xrightarrow{s'} \begin{matrix} \bullet & \leftrightarrow & \bullet \\ \updownarrow & 0 & \updownarrow \\ \bullet & \leftrightarrow & \bullet \end{matrix}, \qquad \updownarrow \xrightarrow{s'} \begin{matrix} \updownarrow & \bullet & \updownarrow \\ \updownarrow & 0 & \updownarrow \\ \updownarrow & \bullet & \updownarrow \end{matrix}, \qquad \leftrightarrow \xrightarrow{s'} \begin{matrix} \leftrightarrow & \leftrightarrow & \leftrightarrow \\ \bullet & 0 & \bullet \\ \leftrightarrow & \leftrightarrow & \leftrightarrow \end{matrix}.$$

Toy case 2 : Sierpiński carpet.

$$
\begin{array}{ccccccccc}
\bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet \\
\updownarrow & & \updownarrow & \bullet & & \bullet & \updownarrow & & \updownarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet \\
\updownarrow & \bullet & \updownarrow & & & & \updownarrow & \bullet & \updownarrow \\
\updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow \\
\updownarrow & \bullet & \updownarrow & & & & \updownarrow & \bullet & \updownarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet \\
\updownarrow & & \updownarrow & \bullet & & \bullet & \updownarrow & & \updownarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet
\end{array}
$$

Toy case 2 : Sierpiński carpet.

$$
\begin{array}{cccc}
a_1 & a_2 & a_3 & a_4 \\
b_1 & b_2 & b_3 & b_4 \\
c_1 & c_2 & c_3 & c_4 \\
d_1 & d_2 & d_3 & d_4
\end{array}
\longleftrightarrow
\begin{array}{ccccccccc}
\bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet \\
\updownarrow & & \updownarrow & \bullet & & \bullet & \updownarrow & & \updownarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet \\
\updownarrow & \bullet & \updownarrow & & & & \updownarrow & \bullet & \updownarrow \\
\updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow \\
\updownarrow & \bullet & \updownarrow & & & & \updownarrow & \bullet & \updownarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet \\
\updownarrow & & \updownarrow & \bullet & & \bullet & \updownarrow & & \updownarrow \\
\bullet & \leftrightarrow & \bullet & \leftrightarrow & \leftrightarrow & \leftrightarrow & \bullet & \leftrightarrow & \bullet
\end{array}
\quad
\begin{array}{ccccccccc}
a_1 & a_2 & a_2 & a_3 & a_3 & a_3 & a_3 & a_4 & a_4 \\
b_1 & & b_2 & \star & & \star & b_3 & & b_4 \\
b_1 & b_2 & b_2 & b_3 & b_3 & b_3 & b_3 & b_4 & b_4 \\
c_1 & \star & c_2 & & & & c_3 & \star & c_4 \\
c_1 & & c_2 & & & & c_3 & & c_4 \\
c_1 & \star & c_2 & & & & c_3 & \star & c_4 \\
c_1 & c_2 & c_2 & c_3 & c_3 & c_3 & c_3 & c_4 & c_4 \\
d_1 & & d_2 & \star & & \star & d_3 & & d_4 \\
d_1 & d_2 & d_2 & d_3 & d_3 & d_3 & d_3 & d_4 & d_4
\end{array}
$$

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  - Use the substitution shown above to simulate arbitrarily big patterns of a $\mathbb{Z}^2$-subshift
  - $\mathtt{DP}(\mathbb{Z}^2)$ is reduced to the domino problem in the carpet.

It only remains to show that we can simulate substitutions with local rules.

Toy case 2 : Sierpiński carpet and Mozes

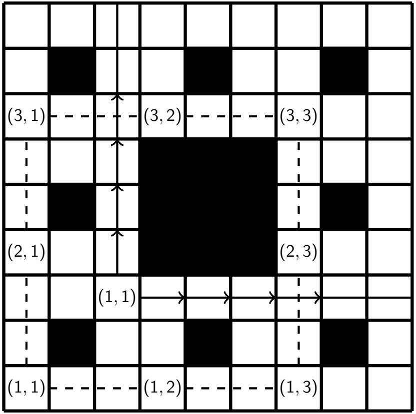
We need to prove a modified version of Mozes' theorem :

Theorem : Mozes.

The subshifts generated by $\mathbb{Z}^2$-substitutions are sofic (are the image of an SFT under a cellular automaton)

We can prove a similar version for some Y-based subshifts. Among them the Sierpiński carpet.

Toy case 2 : Sierpiński carpet and Mozes



Conclusion

We can generalize the ideas in the previous toy problems to attack classes of substitutions :

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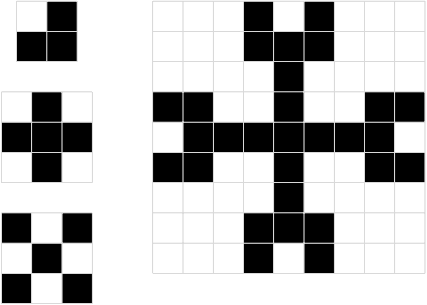
Bounded Connectivity



Decidable domino problem

Conclusion

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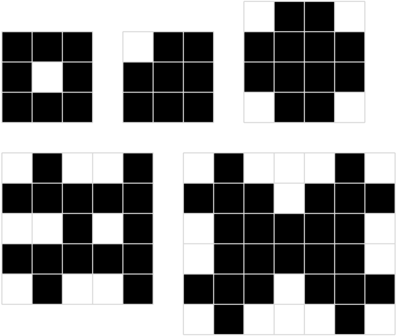
Strong grid



Undecidable domino problem

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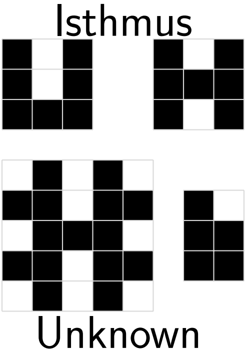

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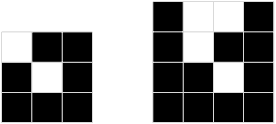
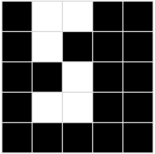
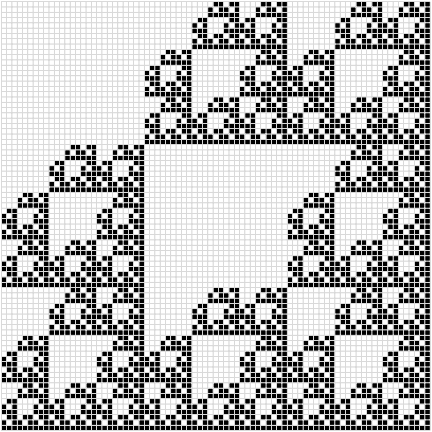
Weak grid





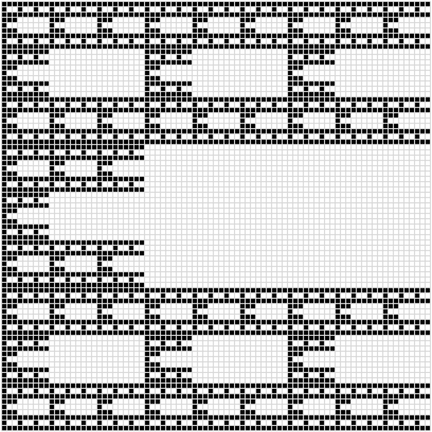
Unknown

Weak grid



We got some ideas of how it might be...

Isthmus



We don't know anything about this one.

Conclusion

And about the Hausdorff dimension ?...

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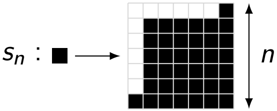

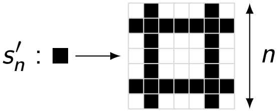

There is no threshold.

Thank you for your attention !