Subshifts in groups: From square-free words on graphs to aperiodic subshifts

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October, 2015

Subshifts in $\mathbb{Z}$

$\mathbb{Z}$-Subshifts

- $\mathcal{A}$ is a finite alphabet. Ex : $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^{\mathbb{Z}}$ is the set of functions from $\mathbb{Z}$ to $\mathcal{A}$
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Alternative definition : $\mathbb{Z}$-subshift

X is a $\mathbb{Z}$-subshift if and only if it can be defined as the set of bi-infinite words which avoid a set forbidden words : $\exists \mathcal{F} \subset \mathcal{A}^*$ such that :

$$X = X_{\mathcal{F}} := \{x \in \mathcal{A}^{\mathbb{Z}} \mid \forall w \in \mathcal{F} : w \not\sqsubset x\}.$$

$\mathbb{Z}$-Subshift examples

Example : full shift. Let $\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \emptyset$. Then $X_{\mathcal{F}} = \mathcal{A}^{\mathbb{Z}}$ is the set of all bi-infinite words.

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Example : Fibonacci shift. Let $\mathcal{A} = \{0, 1\}$ and $\mathcal{F} = \{11\}$. Then X_{Fib} is the set of all bi-infinite words which have no pairs of consecutive 1's.

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$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \ldots$$

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Example : Thue-Morse subshift Consider the morphism $0 \to 01$ and $1 \to 10$. The Thue-Morse subshift $X_{\mathtt{TM}}$ is the set of bi-infinite sequences such that every subword appears as a substring in some iteration of the morphism.

$$x = \dots 10100110010110100101100110100 1 \cdots \in X_{\mathtt{TM}}$$

Periodicity

Definition : periodic point

We say $x \in X$ is periodic if there exists $z \in \mathbb{Z} \setminus \{0\}$ such that :

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Definition : aperiodic $\mathbb{Z}$-subshift

We say $X \subset \mathcal{A}^{\mathbb{Z}}$ is aperiodic if for every configuration $x \in X$ then x is not periodic.

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Example : Thue-Morse subshift The Thue-Morse subshift is indeed aperiodic !

Aperiodicity of the Thue-Morse subshift

Recall the Thue-Morse morphism that acts by $0 \to 01$ and $1 \to 10$.

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The Thue-Morse sequence is cube-free.

Suppose there is $x \in X_{\mathtt{TM}}$ and $z \in \mathbb{Z} \setminus \{0\}$ such that $\sigma_z(x) = x$. Then $x|_{\{1,2,\ldots,3|z|\}}$ is a cube-word, therefore not contained in any iteration of the morphism. This yields a contradiction.

Subshifts of finite type.

- What about if we only consider local rules ?

Definition : subshift of finite type.

A $\mathbb{Z}$-subshift is of finite type (SFT) if it can be defined by a finite set $\mathcal{F}$ of forbidden words.

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- What can we say about periodicity ? Is the Thue-Morse subshift of finite type ?

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The set of configurations of a $\mathbb{Z}$-SFT can be characterized as the set of bi-infinite walks in a finite graph.

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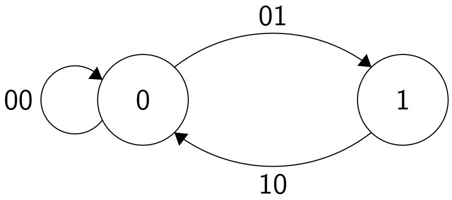

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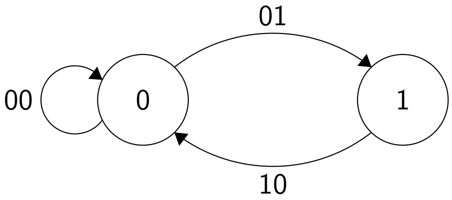

As the graph is finite, an SFT X is non-empty if and only if the graph contains a cycle, thus every $\mathbb{Z}$-SFT contains periodic configurations !

Therefore, the Thue-Morse subshift is not of finite type.

Subshifts in $\mathbb{Z}^2$

$\mathbb{Z}^2$-Subshifts

Now configurations are on the plane.

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Instead of a set of forbidden words, we have a set of forbidden patterns. In the same way as in $\mathbb{Z}$, a $\mathbb{Z}^2$-subshift is of finite type if it is defined by a finite set of forbidden patterns.

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Are there any aperiodic $\mathbb{Z}^2$-SFTs ?

Example in $\mathbb{Z}^2$: Fibonacci shift

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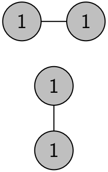
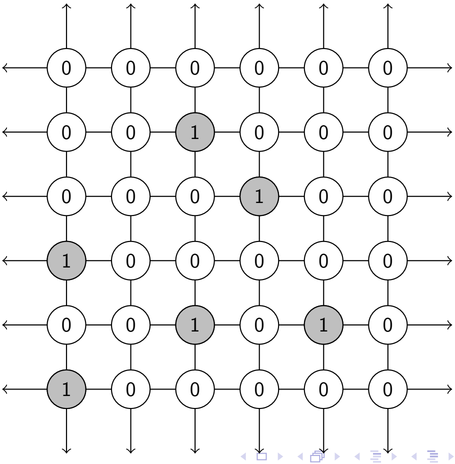
$\mathcal{F} =$ 



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Theorem[Berger 1966]

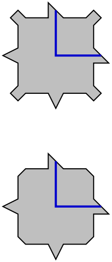
Wang's conjecture is FALSE

His construction encodes a Turing machine using an alphabet of size 20426.

His proof was later simplified by Robinson[1971].

Robinson tileset

The Robinson tileset, where tiles can be rotated.

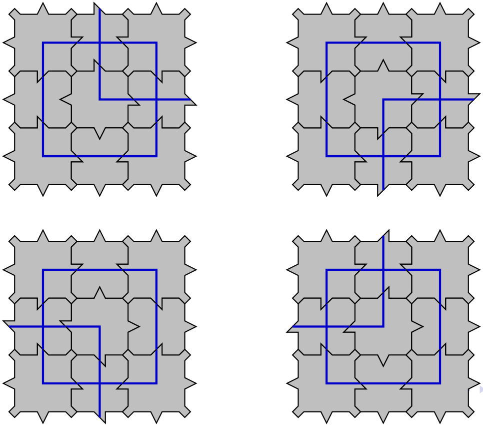


Existence of a valid tiling

Proposition

Robinson's tileset admits at least one valid tiling.

Macro-tiles of level 1.



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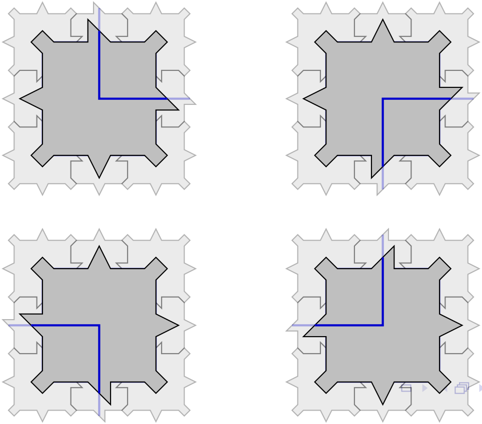
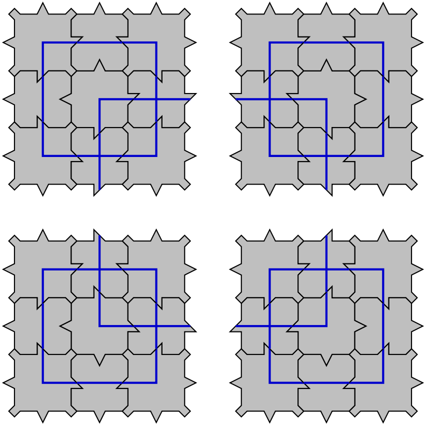


From macro-tiles of level 1 to macro-tiles of level 2



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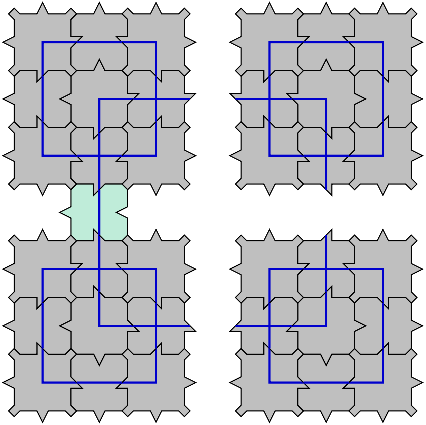

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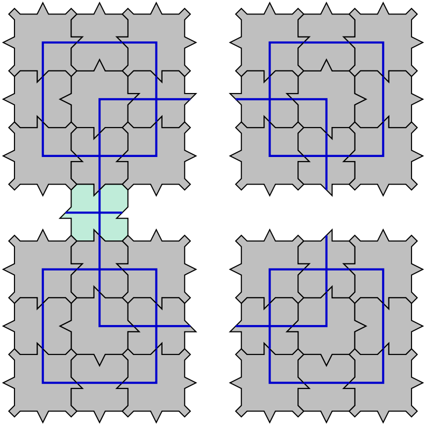

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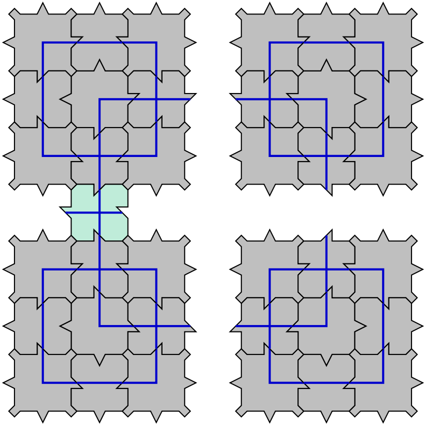

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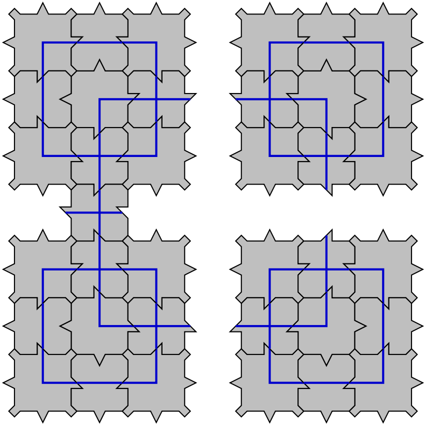

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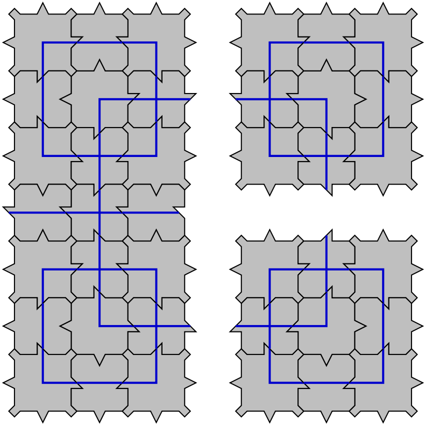

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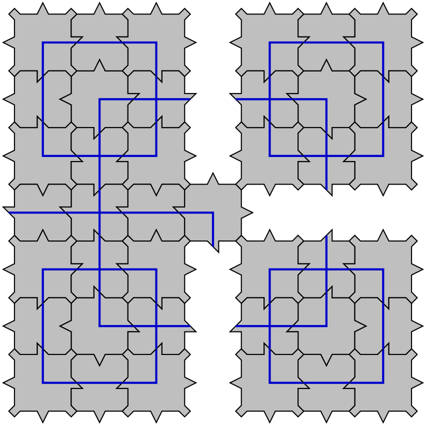

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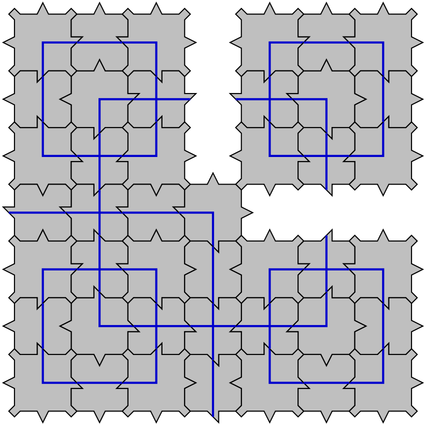

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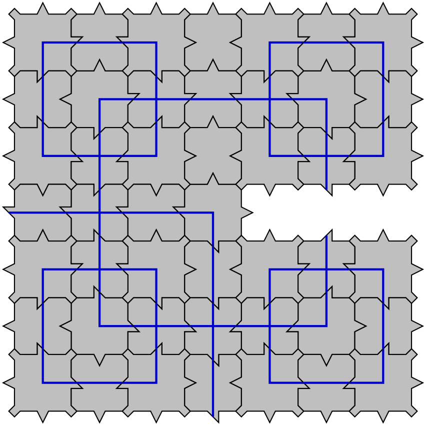

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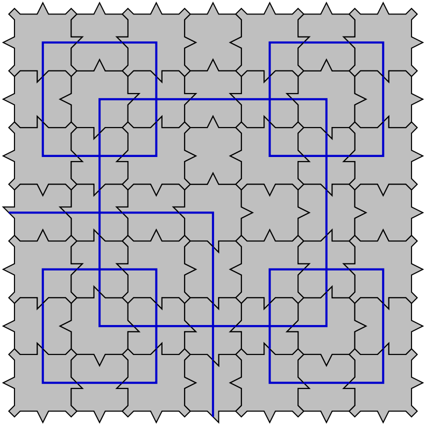

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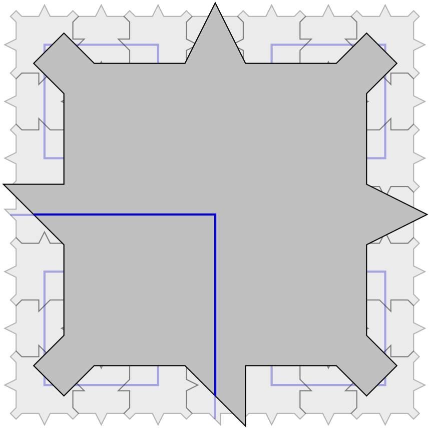
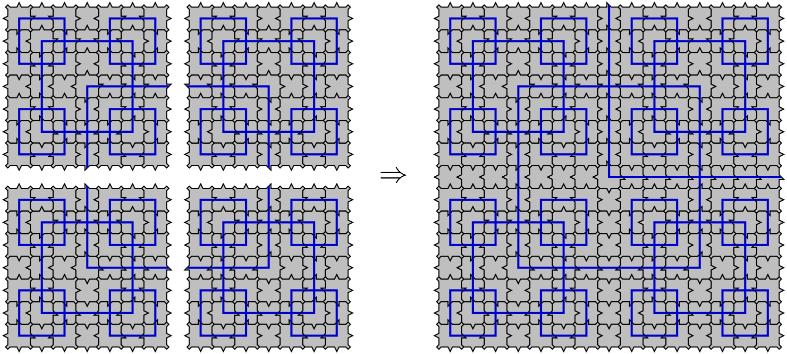


From macro-tiles of level n to macro-tiles of level $n+1$



One can thus construct a valid tiling by a compacity argument.

This valid tiling is aperiodic

Proposition

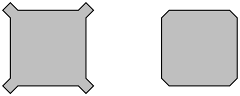
The valid tiling x obtained by compactness is aperiodic.

Proof :

- Centers of macro-tiles of level n are located a $2^{n+1}\mathbb{Z} \times 2^{n+1}\mathbb{Z}$-lattice.
- Suppose x admits a direction of periodicity $\overrightarrow{u}$.
- Then there exists an integer n s.t. $2^{n+1} > \|\overrightarrow{u}\|$.
- Thus a macro-tile of level n overlaps with its translation.
- $\Rightarrow$ contradiction.

All valid tilings are aperiodic (I)

The two forms in Robinson tileset, cross (bumpy corners) and arms (dented corners).



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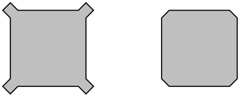
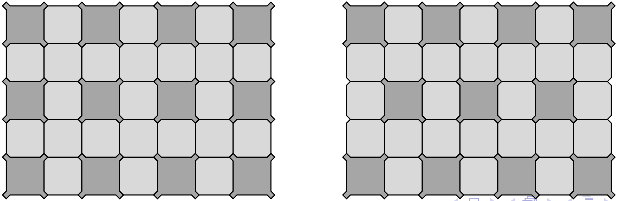
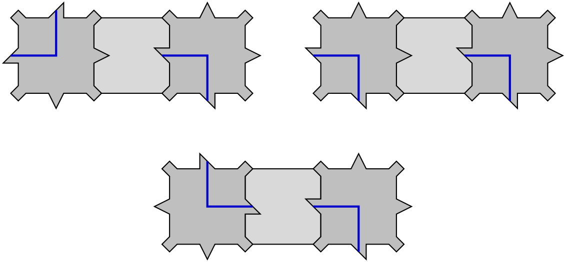


Obviously, two crosses cannot be in contact (neither through an edge nor a vertex) hence a cross must be surrounded by eight arms.



All valid tilings are aperiodic (II)

You cannot have things like



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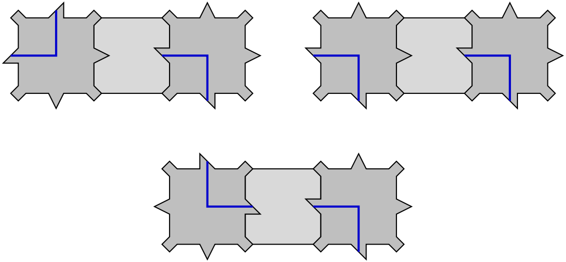

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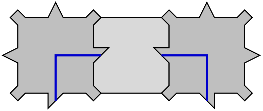

All valid tilings are aperiodic (II)

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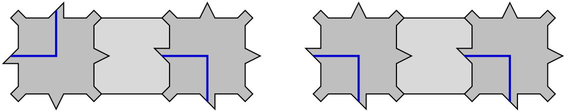

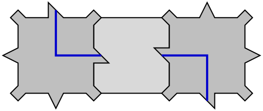

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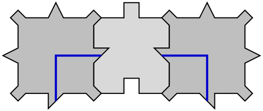
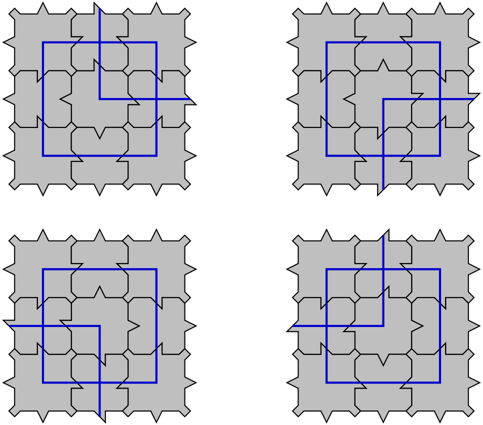


All valid tilings are aperiodic (III)

So each  is part of a macro tile of level 1



that behaves like a big , and so on...

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Therefore, any configuration in the Robinson tiling contains arbitrary level macro-tiles and the previous argument shows they cannot have any periodic behavior.

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So far, we have shown...

- $\mathbb{Z}$-SFTs are never aperiodic.
- There exist aperiodic $\mathbb{Z}^2$-SFTs. (1966 Berger, 1971 Robinson, 1996 Kari)

Subshifts in finitely generated groups

Finitely generated groups

Now configurations are on a group G.

- $\mathcal{A}^G$ is the set of functions from G to $\mathcal{A}$
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the shift action given by :

$$\sigma_g(x)_h = x_{g^{-1}h}.$$

- G-subshifts are defined analogously.

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Are there any aperiodic G-SFTs ?

There is no general characterization of which groups admit aperiodic SFTs.

Some recent partial results

- There are weakly aperiodic SFTs in Baumslag Solitar groups (2013 Aubrun-Kari)
- There are aperiodic SFTs in the Heisenberg group (2014 Sahin-Schraudner)
- The existence of an aperiodic SFT in G implies that G is one ended (2014 Cohen)
- The existence of an aperiodic SFT is a quasi-isometry invariant for finitely presented torsion-free groups. (2014 Cohen)
- A recursively presented group which admits an aperiodic SFT has decidable word problem (2015 Jeandel)

Some recent partial results

It is even hard to come up with examples of aperiodic subshifts in general groups if we retreat the finite type property of the list of forbidden patterns.

Question by Glasner and Uspenskij 2009

Is there any countable group which does not admit any aperiodic subshift on a two symbol alphabet ?

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Theorem by Gao, Jackson and Seward 2009

No. All do.

And the proof is a quite technical construction.

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We will present here a slightly different version which uses a generalization of square free words. It will only work for finitely generated groups and it will use a larger alphabet.

Lovász Local Lemma

Lovász Local Lemma (Asymmetrical version)

Let $\mathscr{A} := \{A_1, A_2, \ldots, A_n\}$ be a finite collection of measurable sets in a probability space $(X, \mathbb{P}, \mathcal{B})$. For $A \in \mathscr{A}$, let $\Gamma(A)$ be the subset of $\mathscr{A}$ such that A is independent of the collection $\mathscr{A} \setminus (\{A\} \cup \Gamma(A))$. Suppose there exists a function $x : \mathscr{A} \to (0, 1)$ such that :

$$\forall A \in \mathscr{A} : \mathbb{P}(A) \leq x(A) \prod_{B \in \Gamma(A)} (1 - x(B))$$

then the probability of avoiding all events in $\mathscr{A}$ is positive, in particular :

$$\mathbb{P}\left(X \setminus \bigcup_{i=1}^{n} A_i\right) \geq \prod_{A \in \mathscr{A}} (1 - x(A)) > 0.$$

Back to Thue-Morse

Recall the Thue-Morse morphism $0 \to 01$ and $1 \to 10$.

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It is not hard to modify it so that it avoids squares.

$$0 \to 01210$$
$$1 \to 12021$$
$$2 \to 20102$$

$$0 \to 01210 \to 0121012021201021202101210\ldots$$

We can interpret this sequence of words as colorings without squares over a path graph.

Square-free vertex coloring

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Let $G = (V, E)$ be a graph. A vertex coloring is a function $x : V \to \mathcal{A}$. We say it is square-free if for every odd-length path $p = v_1 \dots v_{2n}$ then there exists $1 \leq j \leq n$ such that $x(v_j) \neq x(v_{j+n})$.

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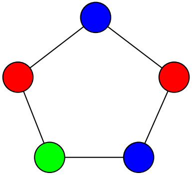

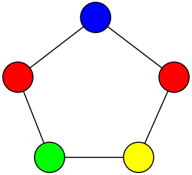

C_5 has a square-free vertex coloring with 4 colors, but not with 3.

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Theorem : Alon, Grytczuk, Haluszczak and Riordan

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It is possible to adapt the proof in order to obtain the following :
Let G be a group which is generated by a finite set S and let $\Gamma(G,S) = (G, \{\{g, gs\}, g \in G, s \in S\})$ be its undirected right Cayley graph.

Theorem

G admits a coloring of its undirected Cayley graph $\Gamma(G,S)$ with $2^{19}|S|^2$ colors.

Proof sketch

- Choose a coloring of the graph uniformly :
  $\forall g \in G, a \in \mathcal{A}, \mathbb{P}(\{x \in \mathcal{A}^{\Gamma(G,S)} \mid x_g = a\}) = 1/|\mathcal{A}|$.

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- The probability of the event : A_p : a configuration has a square path p of length $2n-1$ is $|\mathcal{A}|^{-n}$.
- Setting $x(A_p) = (8|S|^2)^{-n}$ for a path of length $2n-1$ satisfies LLL if $|\mathcal{A}| > 2^{19}|S|^2$.

The construction

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- Factorize g as uwv with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1$.
- If not, let $w = w_1 \dots w_n$ and consider the odd length walk $\pi = v_0 v_1 \dots v_{2n-1}$ on $\Gamma(G, S)$ defined by :

$$v_i = \begin{cases} 1 & \text{if } i = 0 \\ w_1 \dots w_i & \text{if } i \in \{1, \dots, n\} \\ w w_1 \dots w_{i-n} & \text{if } i \in \{n+1, \dots, 2n-1\} \end{cases}$$

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- One can prove that π is a path. and that $x_{v_i} = x_{v_{i+n}}$. Yielding a contradiction.
- Therefore, $g = 1$.

Conclusion

What we have shown here :

Every finitely generated group admits a non-empty aperiodic subshift over (a big) alphabet.

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Actually, we can prove the original result and add some extra spice.

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Every countable group G admits a non-empty aperiodic subshift over $\{0, 1\}$. Moreover, if G is finitely generated and has decidable word problem then the subshift is defined by a recursively enumerable set of forbidden patterns

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Is is possible to use this construction to produce aperiodic SFTs ?

Merci beaucoup pour votre attention !