A short proof of the existence of non-empty strongly aperiodic subshifts over $\{0, 1\}$ in countable groups.

Sebastián Barbieri
LIP, ENS de Lyon

Séminaire LIMD
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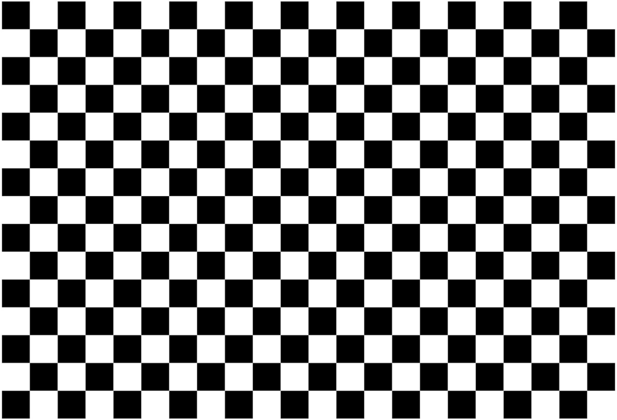
This is a periodic tiling



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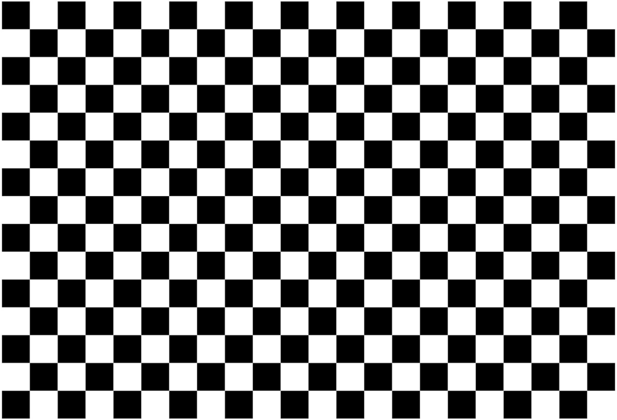
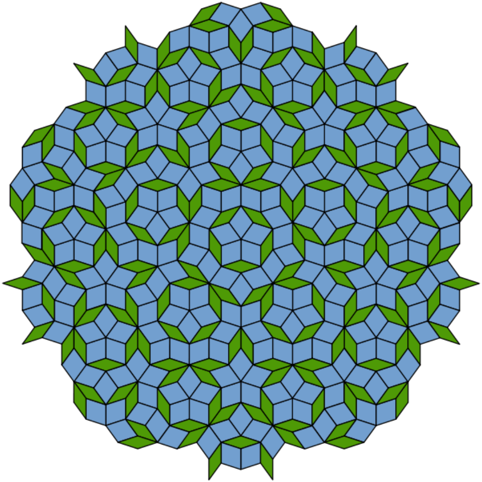


This is BORING.



This is cool !

Cool things are useful in real life !

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A good example of uses of aperiodic tilings.

Subshifts

G-Subshifts

- G is a countable group.
- $\mathcal{A}$ is a finite alphabet. Ex : $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of configurations, $x : G \to \mathcal{A}$
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the left shift action given by :

$$\sigma(h, x)_g := \sigma_h(x)_g = x_{h^{-1}g}.$$

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Alternatively : $\exists \mathcal{F} \subset \bigcup_{F \subset G, |F| < \infty} \mathcal{A}^F$ such that :

$$X = X_{\mathcal{F}} := \mathcal{A}^G \setminus \bigcup_{g \in G, p \in \mathcal{F}} \sigma_g([p])$$

Example in $\mathbb{Z}^2$: Fibonacci shift

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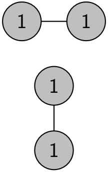
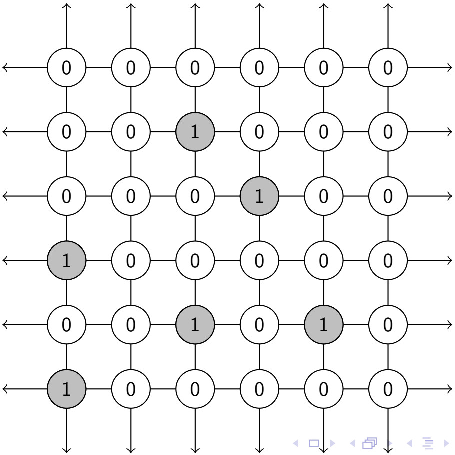
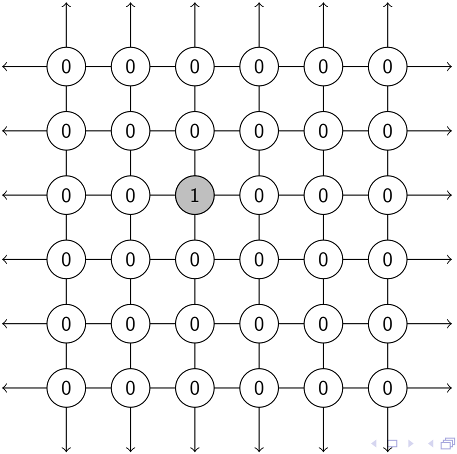
$\mathcal{F} =$ 

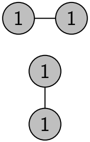


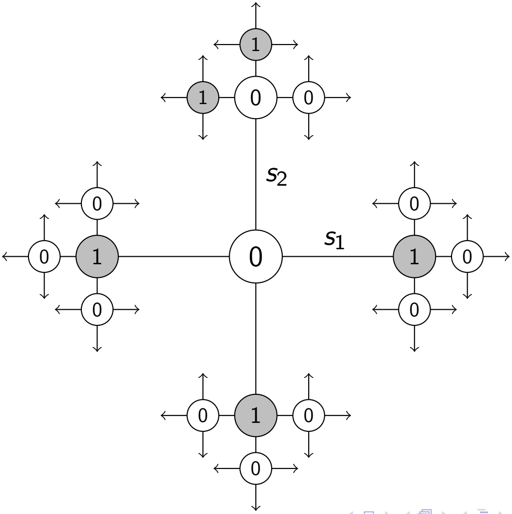
Example : One-or-less subshift.

$$X_{\leq 1} := \{x \in \{0,1\}^{\mathbb{Z}^d} \mid |\{z \in \mathbb{Z}^d : x_z = 1\}| \leq 1\}.$$



Fibonacci in F_2.

$\mathcal{F} =$ 



Classes of subshifts.

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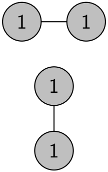
Definition : G-Effectively closed subshift.

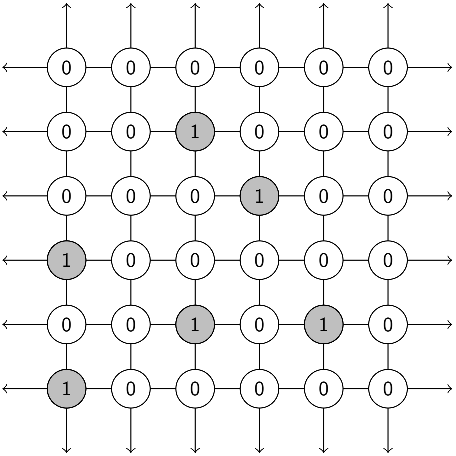
A subshift is G-*effectively closed* if it can be defined by a set of forbidden patterns $\mathcal{F}$ which is recognizable by a Turing machine with oracle the word problem of G.

Example SFT

Example : The Fibonacci shift is an $\mathbb{Z}^2$-SFT.

$\mathcal{F} =$





Example : The one-or-less subshift is sofic but not an SFT.

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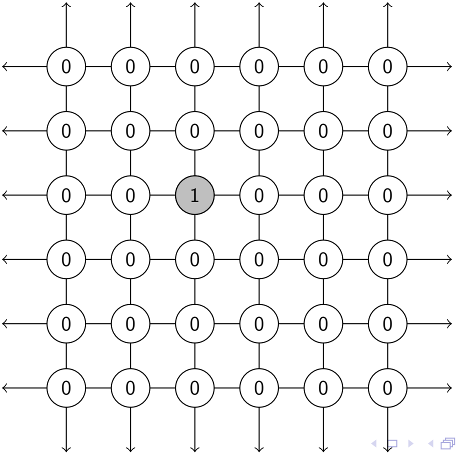
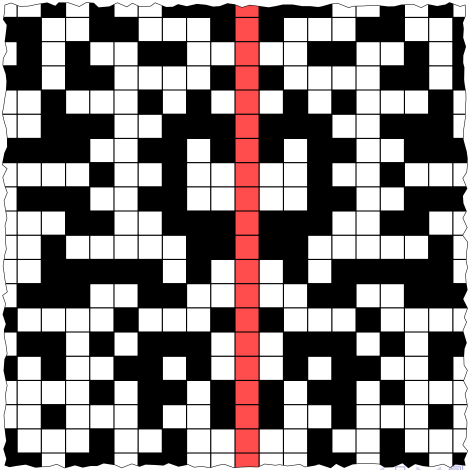


Example of effectively closed subshift

Example : The mirror shift is effectively closed but not sofic.



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We say $X \subset \mathcal{A}^G$ is weakly aperiodic if the orbit of every configuration $x \in X$ is infinite.

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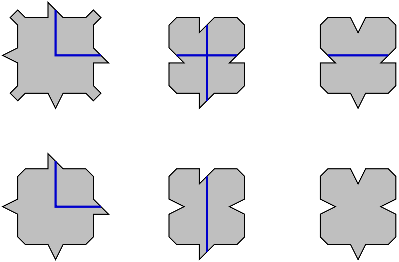
Obviously, we are interested in non-empty aperiodic subshifts.

Example of strongly aperiodic $\mathbb{Z}^2$-SFT : Robinson tileset

The Robinson tileset, where tiles can be rotated.



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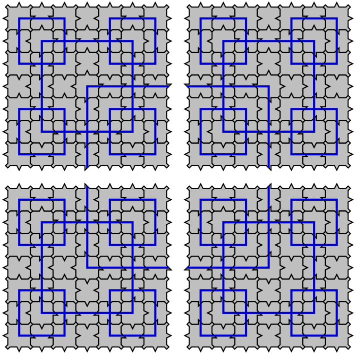
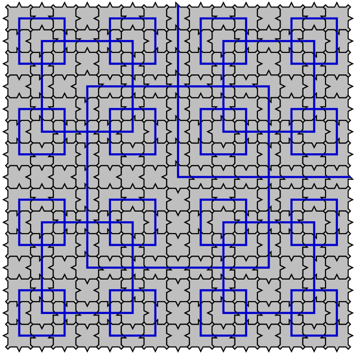


$\Rightarrow$



Some recent results

- There are weakly aperiodic SFTs in Baumslag Solitar groups (2013 Aubrun-Kari)
- There are strongly aperiodic SFTs in the Heisenberg group (2014 Sahin-Schraudner)
- The existence of a strongly aperiodic SFT in G implies that G is one ended (2014 Cohen)
- The existence of a non-empty strongly aperiodic SFT is a quasi-isometry invariant for finitely presented torsion-free groups. (2014 Cohen)
- A recursively presented group which admits a non-empty strongly aperiodic SFT has decidable word problem (2015 Jeandel)

Some recent partial results

It is not obvious to come up with examples of aperiodic subshifts in general groups even if no restrictions are supposed on the list of forbidden patterns.

Question by Glasner and Uspenskij 2009

Is there any countable group which does not admit any non-empty strongly aperiodic subshift on a two symbol alphabet ?

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Theorem by Gao, Jackson and Seward 2009

No. All do.

And their proof is a quite technical construction.

A new short proof

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Theorem by Aubrun, B, Thomassé

No. All do.

But now the proof is short. It uses the asymmetrical version of Lovász Local Lemma.

Lovász Local Lemma

Lovász Local Lemma (Asymmetrical version)

Let $\mathscr{A} := \{A_1, A_2, \ldots, A_n\}$ be a finite collection of measurable sets in a probability space $(X, \mu, \mathcal{B})$. For $A \in \mathscr{A}$, let $\Gamma(A)$ be the smallest subset of $\mathscr{A}$ such that A is independent of the collection $\mathscr{A} \setminus (\{A\} \cup \Gamma(A))$. Suppose there exists a function $x : \mathscr{A} \to (0, 1)$ such that :

$$\forall A \in \mathscr{A} : \mu(A) \leq x(A) \prod_{B \in \Gamma(A)} (1 - x(B))$$

then the probability of avoiding all events in $\mathscr{A}$ is positive, in particular :

$$\mu\left(X \setminus \bigcup_{i=1}^{n} A_i\right) \geq \prod_{A \in \mathscr{A}} (1 - x(A)) > 0.$$

Lovász Local Lemma applied to subshifts

A sufficient condition for being non-empty

Let G a countable group and $X \subset \mathcal{A}^G$ a subshift defined by the set of forbidden patterns $\mathcal{F} = \bigcup_{n \geq 1} \mathcal{F}_n$, where $\mathcal{F}_n \subset \mathcal{A}^{S_n}$. Suppose that there exists a function $x : \mathbb{N} \times G \to (0,1)$ such that :

$$\forall n \in \mathbb{N}, g \in G,\ \mu(A_{n,g}) \leq x(n,g) \prod_{\substack{gS_n \cap hS_k \neq \emptyset \\ (k,h) \neq (n,g)}} (1 - x(k,h)),$$

where $A_{n,g} = \left\{ x \in \mathcal{A}^G : x|_{gS_n} \in \mathcal{F}_n \right\}$ and μ is any Bernoulli probability measure on $\mathcal{A}^G$. Then the subshift X is non-empty.

Proof of the theorem

We say $x \in \{0,1\}^G$ has *the distinct neighborhood property* if for every $h \in G \setminus \{1_G\}$ there exists a finite subset $T \subset G$ such that :

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Proposition

If x has the distinct neighborhood property then $\overline{orb_\sigma(x)}$ is strongly aperiodic.

Proof of the theorem

It suffices to show that there is $x \in \{0, 1\}^G$ with the distinct neighborhood property.

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Ingredients

- A constant $C \in \mathbb{N}$.
- An enumeration $s_1, s_2, \ldots$ of G.
- $(T_i)_{i\in\mathbb{N}}$ a sequence of finite subsets of G such that for every $i \in \mathbb{N}$, $T_i \cap s_i T_i = \emptyset$ and $|T_i| = C \cdot i$.
- The uniform Bernoulli measure μ
- $\mathscr{A} := \{A_{n,g}\}_{n\geq 1, g\in G}$
- $A_{n,g} = \{x \in \{0,1\}^G \mid x|_{gT_n} = x|_{gs_nT_n}\}$
- $x(A_{n,g}) := 2^{-\frac{Cn}{2}}$

Proof : On the blackboard.

Aftermath

We have shown :

Theorem

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But we can show something more :

Theorem (Aubrun, B, Thomassé)

Every finitely generated group G has a non-empty, G-effectively closed strongly aperiodic subshift.

Square-free vertex coloring

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Let $G = (V, E)$ be a graph. A vertex coloring is a function $x : V \to \mathcal{A}$. We say it is square-free if for every odd-length path $p = v_1 \dots v_{2n}$ then there exists $1 \leq j \leq n$ such that $x(v_j) \neq x(v_{j+n})$.

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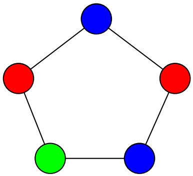

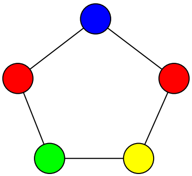

C_5 has a square-free vertex coloring with 4 colors, but not with 3.

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It is possible to adapt the proof in order to obtain the following :
Let G be a group which is generated by a finite set S and let $\Gamma(G,S) = (G, \{\{g, gs\}, g \in G, s \in S\})$ be its undirected right Cayley graph.

Theorem

G admits a coloring of its undirected Cayley graph $\Gamma(G,S)$ with $2^{19}|S|^2$ colors.

The proof idea

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- If not, let $w = w_1 \dots w_n$ and consider the odd length walk $\pi = v_0 v_1 \dots v_{2n-1}$ on $\Gamma(G, S)$ defined by :

$$v_i = \begin{cases} 1_G & \text{if } i = 0 \\ w_1 \dots w_i & \text{if } i \in \{1, \dots, n\} \\ w w_1 \dots w_{i-n} & \text{if } i \in \{n+1, \dots, 2n-1\} \end{cases}$$

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- Therefore, $g = 1_G$.

Applications

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Putting it together with Jeandel's result we get :

Theorem

Let G be a recursively presented group. There exists a non-empty $\mathbb{Z}$-effectively closed strongly aperiodic G-subshift if and only if the word problem of G is decidable.

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Theorem (B, Sablik (hopefully available on 2016))

Let G be a finitely generated group with decidable word problem. Then $\mathbb{Z}^2 \rtimes G$ admits a non-empty strongly aperiodic SFT.

The writing of this is still in progress !

Uniform density subshifts

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A G-subshift over $\{0,1\}$ has *uniform density* $\alpha \in [0,1]$ if for every configuration $x \in X$ and for every sequence $(g_n)_{n\in\mathbb{N}}$ of elements in G, one has

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Example : Sturmian subshift

The Sturmian $\mathbb{Z}$-subshift of slope α has uniform density α.

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Given a finitely generated group G, a finite set of generators S and $\alpha \in (0, 1)$. Does a subshift $X_\alpha \subset \{0, 1\}^G$ with uniform density α exist ?

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Yes, if G is infinite and has subexponential growth.

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Given a finitely generated group G, a finite set of generators S and $\alpha \in (0,1)$. Does a subshift $X_\alpha \subset \{0,1\}^G$ with uniform density α exist ?

Theorem (Aubrun, B, Thomassé)

Yes, if G is infinite and has subexponential growth.

More precisely we show :

Theorem

Let G be an infinite and finitely generated group and $\alpha \in [0,1]$. There is a non-empty subshift $X_\alpha \subset \{0,1\}^G$ such that for any $x \in X_\alpha$ and Følner sequence $(F_n)_{n\in\mathbb{N}}$

$$\lim_{n\to\infty} dens(1, F_n, x) = \alpha.$$

Proof

Definition

Let F, K be finite subsets of G

- $Int(F, K) = \{g \in F \mid \forall k \in K, gk \in F\}$
- $\partial_K(F) = F \setminus Int(F, K)$

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Let X_α be given by the following forbidden patterns : $P \in \{0,1\}^F$ is forbidden if and only if the following condition is not satisfied :

$$\frac{|\partial_{B(1_G,5^n)}F|}{|F|} < \frac{1}{2n} \implies |dens(1,P) - \alpha| \leq \frac{1}{n}.$$

X_α clearly satisfies the property. It suffices to show that it is non-empty.

An ingredient for the proof

Definition

Let (X, d) be a metric space. We say $F \subset G$ is *r-covering* if for each $x \in G$ there is $y \in F$ such that $d(x, y) \leq r$. We say F is *s-separating* if for each $x \neq y \in F$ then $d(x, y) > s$

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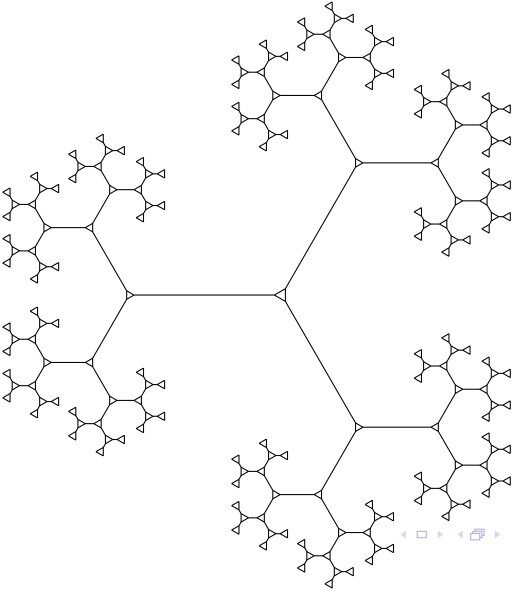
Proposition

If X is countable, then for any $r \in \mathbb{R}$ there exists $Y \subset X$ such that Y is both r-separating and r-covering.

Example : 2-covering and 2-separating set in $PSL(\mathbb{Z}, 2)$



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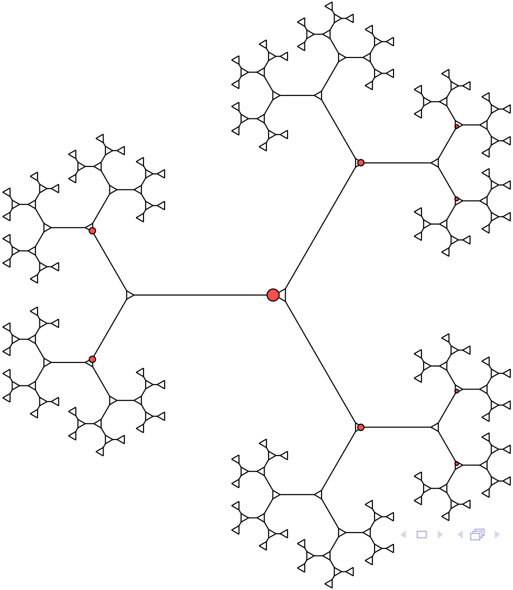

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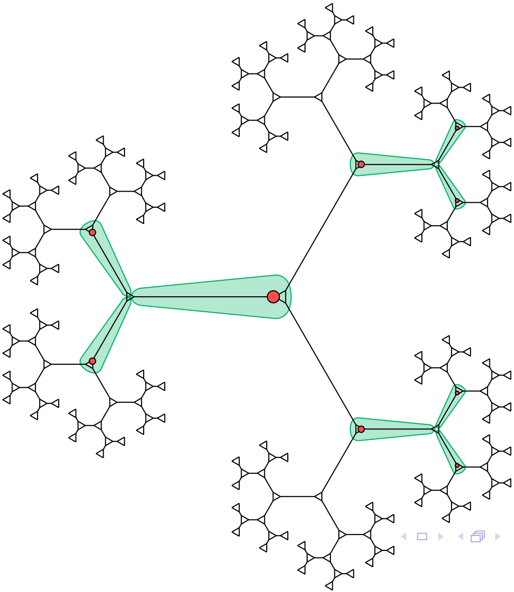

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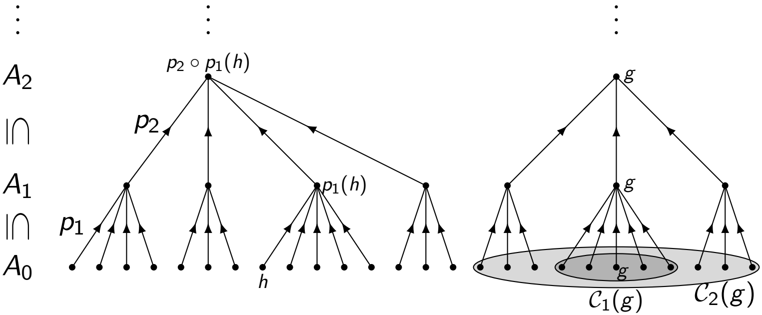
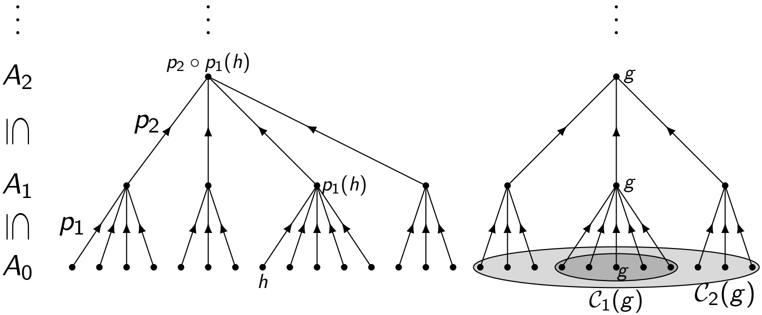


A_2
$\cap$
A_1
$\cap$
A_0
$p_2 \circ p_1(h)$
p_2
p_1
$p_1(h)$
h
g
g
g
$\mathcal{C}_1(g)$
$\mathcal{C}_2(g)$

Proof



We can bound the size of a cluster of level n :

$$B(g,n) \subset \mathcal{C}_n(g) \leq B(g, \frac{1}{2}(5^n - 1)).$$

Proof

- Finally, consider a function $\phi : G \to \mathbb{N}$ satisfying that if $\mathcal{C}_n(g) = \mathcal{C}_n(h)$ then $\phi(g) \neq \phi(h)$ and all integers between $\phi(g)$ and $\phi(h)$ belong to $\mathcal{C}_n(g)$.

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- Let $x \in \{0,1\}^G$ defined by $x_g = w_{\phi(g)}$ where $w : \mathbb{N} \to \{0,1\}$ is a Sturmian word of slope α (or a periodic configuration if $\alpha \in \mathbb{Q}$).

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- $x \in X_\alpha$ by a straightforward calculation.

Final remarks

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- We can also realize every density in free groups
- Hope : reduce the factor complexity of this construction to realize entropies using $\bigcup_{\alpha<h} X_\alpha$.

Merci beaucoup de votre attention !