# Shift spaces on groups: computability and dynamics

Calculabilité et dynamique des sous-décalages sur des groupes

Sebastián Andrés **Barbieri Lemp**

LIP, ENS de Lyon

Soutenance de Thèse
28 Juin 2017

# Motivation

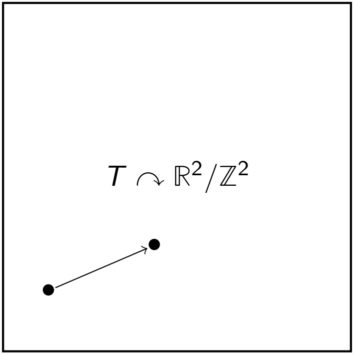
Consider $T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \mod 1$$

# Motivation

Consider $T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \mod 1$$



# Motivation

Consider $T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \mod 1$$

# Motivation

# Motivation

$T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \mod 1$$

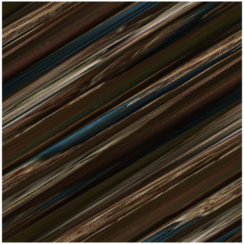
$\longrightarrow T \longrightarrow$

# Motivation

$T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \mod 1$$

$\xrightarrow{\quad T^2 \quad}$

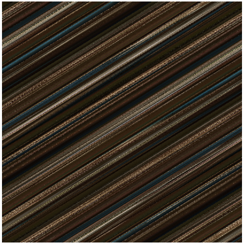


# Motivation

$T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \mod 1$$

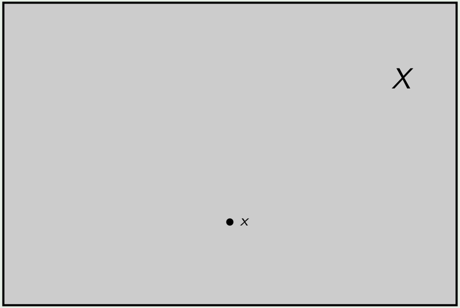
$\xrightarrow{\quad T^3 \quad}$



# What's a shift space?

A dynamical system might be complicated. A good idea is to code its trajectories using a partition.

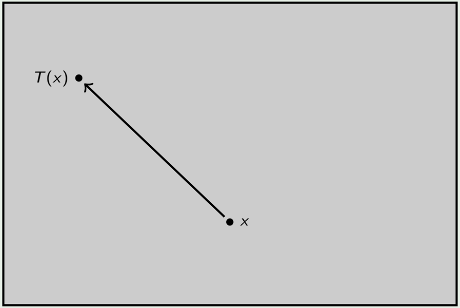
**A $\mathbb{Z}$-action by homeomorphisms.**



# What's a shift space?

A dynamical system might be complicated. A good idea is to code its trajectories using a partition.

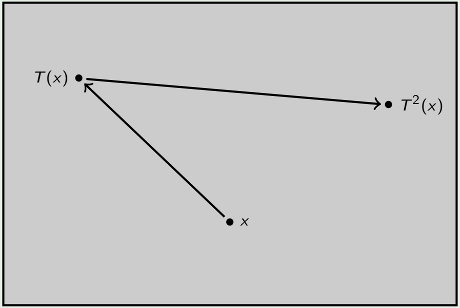
**A $\mathbb{Z}$-action by homeomorphisms.**



# What's a shift space?

A dynamical system might be complicated. A good idea is to code its trajectories using a partition.

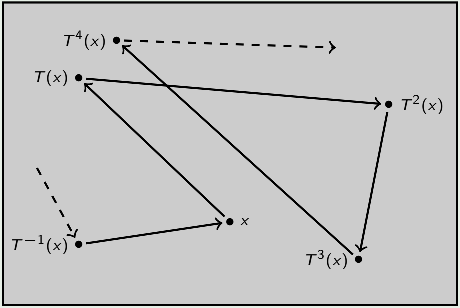
## A $\mathbb{Z}$-action by homeomorphisms.



# What's a shift space?

A dynamical system might be complicated. A good idea is to code its trajectories using a partition.

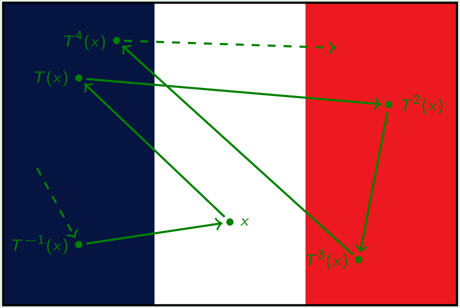
## A $\mathbb{Z}$-action by homeomorphisms.

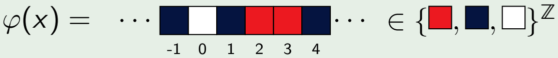


# What's a shift space?

A dynamical system might be complicated. A good idea is to code its trajectories using a partition.

## A $\mathbb{Z}$-action by homeomorphisms.

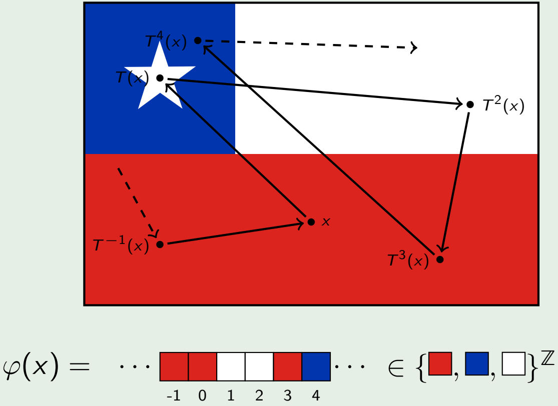


# What's a shift space?

A dynamical system might be complicated. A good idea is to code its trajectories using a partition.

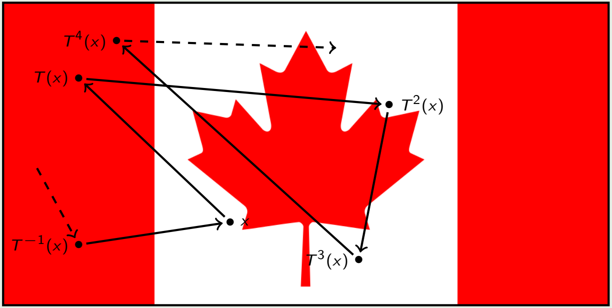
## A $\mathbb{Z}$-action by homeomorphisms.

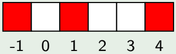


# What's a shift space?

A dynamical system might be complicated. A good idea is to code its trajectories using a partition.

## A $\mathbb{Z}$-action by homeomorphisms.




$\varphi(x) = \cdots$ 

$\cdots \in \{$  $\}^{\mathbb{Z}}$

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$.
We are interested in the system where both $S$ and $T$ act over $X$.

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$.
We are interested in the system where both $S$ and $T$ act over $X$.

▷ $T$ is an odometer acting on $\{0,1\}^{\mathbb{Z}}$: $\quad T \approx +1$.

$$x \ = \ldots 01000100010.0100101010010 \ldots$$

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$.
We are interested in the system where both $S$ and $T$ act over $X$.

▷ $T$ is an odometer acting on $\{0,1\}^{\mathbb{Z}}$: $\quad T \approx +1.$

$$T\ (x) = \ldots 01000100011.0100101010010\ldots$$

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$.
We are interested in the system where both $S$ and $T$ act over $X$.

- $\triangleright$ $T$ is an odometer acting on $\{0,1\}^{\mathbb{Z}}$: $\qquad T \approx +1.$

$$T^2(x) = \ldots 01000100100.0100101010010 \ldots$$

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$.
We are interested in the system where both $S$ and $T$ act over $X$.

▷ $T$ is an odometer acting on $\{0,1\}^{\mathbb{Z}}$: $\qquad T \approx +1.$

$$T^3(x) = \ldots 01000100101.0100101010010 \ldots$$

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$.
We are interested in the system where both $S$ and $T$ act over $X$.

$\triangleright$ $T$ is an odometer acting on $\{0,1\}^{\mathbb{Z}}$: $\quad T \approx +1$.

$$T^4(x) = \ldots 01000100110.0100101010010 \ldots$$

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$. We are interested in the system where both $S$ and $T$ act over $X$.

- $\triangleright$ $T$ is an odometer acting on $\{0,1\}^{\mathbb{Z}}$: $T \approx +1$.

$$T^4(x) = \ldots 01000100110.0100101010010 \ldots$$

- $\triangleright$ $S$ is a shift acting on $\{0,1\}^{\mathbb{Z}}$: $S \approx \times 2$.

$$x = \ldots 01000100010.0100101010010 \ldots$$

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$.
We are interested in the system where both $S$ and $T$ act over $X$.

- $\triangleright$ $T$ is an odometer acting on $\{0,1\}^{\mathbb{Z}}$: $T \approx +1$.

$$T^4(x) = \ldots 01000100110.0100101010010 \ldots$$

- $\triangleright$ $S$ is a shift acting on $\{0,1\}^{\mathbb{Z}}$: $S \approx \times 2$.

$$S\ (x) = \ldots 10001000100.1001010100100 \ldots$$

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$. We are interested in the system where both $S$ and $T$ act over $X$.

- ▷ $T$ is an odometer acting on $\{0,1\}^{\mathbb{Z}}$: $T \approx +1$.

$$T^4(x) = \ldots 01000100110.0100101010010 \ldots$$

- ▷ $S$ is a shift acting on $\{0,1\}^{\mathbb{Z}}$: $S \approx \times 2$.

$$S^2(x) = \ldots 00010001001.0010101001001 \ldots$$

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$.
We are interested in the system where both $S$ and $T$ act over $X$.

- $T$ is an odometer acting on $\{0,1\}^{\mathbb{Z}}$: $T \approx +1$.

$$T^4(x) = \ldots 01000100110.0100101010010\ldots$$

- $S$ is a shift acting on $\{0,1\}^{\mathbb{Z}}$: $S \approx \times 2$.

$$S^3(x) = \ldots 00100010010.0101010010011\ldots$$

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$.
We are interested in the system where both $S$ and $T$ act over $X$.

- $\triangleright$ $T$ is an odometer acting on $\{0,1\}^{\mathbb{Z}}$: $T \approx +1$.

$$T^4(x) = \ldots 01000100110.0100101010010\ldots$$

- $\triangleright$ $S$ is a shift acting on $\{0,1\}^{\mathbb{Z}}$: $S \approx \times 2$.

$$S^4(x) = \ldots 01000100100.1010100100110\ldots$$

# Why a group action?

Let $T, S$ be two homeomorphisms $T : X \to X$ and $S : X \to X$.
We are interested in the system where both $S$ and $T$ act over $X$.

- $\triangleright$ $T$ is an odometer acting on $\{0,1\}^{\mathbb{Z}}$: $\quad T \approx +1.$

$$T^4(x) = \ldots 01000100110.0100101010010\ldots$$

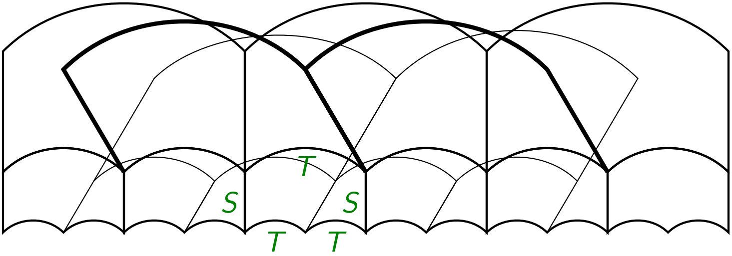
- $\triangleright$ $S$ is a shift acting on $\{0,1\}^{\mathbb{Z}}$: $\quad S \approx \times 2.$

$$S^4(x) = \ldots 01000100100.1010100100110\ldots$$

$$2 \times (x+1) = (2 \times x) + 2 \quad \rightsquigarrow \quad ST = T^2 S$$

# Why a group action?

$$\langle S, T \mid ST = T^2S \rangle \cong BS(1,2)$$



# Context

- $G$ is a finitely generated group.
- $\mathcal{A}$ is a finite alphabet. Ex: $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of configurations, $x : G \to \mathcal{A}$
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the left shift action given by:

$$\sigma(h, x)_g := \sigma^h(x)_g = x_{h^{-1}g}.$$

# Context

- $G$ is a finitely generated group.
- $\mathcal{A}$ is a finite alphabet. Ex: $\mathcal{A} = \{0, 1\}$.
- $\mathcal{A}^G$ is the set of configurations, $x : G \to \mathcal{A}$
- $\sigma : G \times \mathcal{A}^G \to \mathcal{A}^G$ is the left shift action given by:

$$\sigma(h, x)_g := \sigma^h(x)_g = x_{h^{-1}g}.$$

**Definition: full $G$-shift**

The pair $(\mathcal{A}^G, \sigma)$ is called the *full $G$-shift*.

# Example

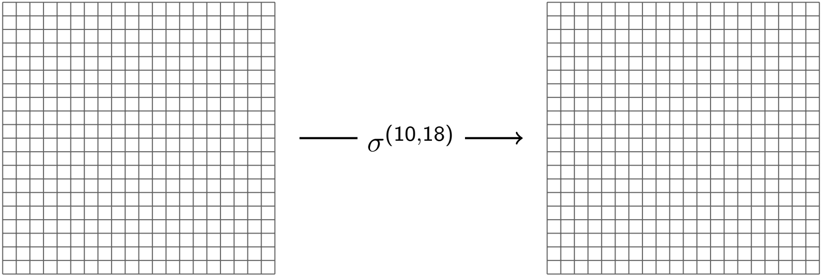



Figure: A random configuration $x \in \{\blacksquare, \square\}^{\mathbb{Z}^2/20\mathbb{Z}^2}$ and its image by $\sigma^{(10,18)}$.

# Example

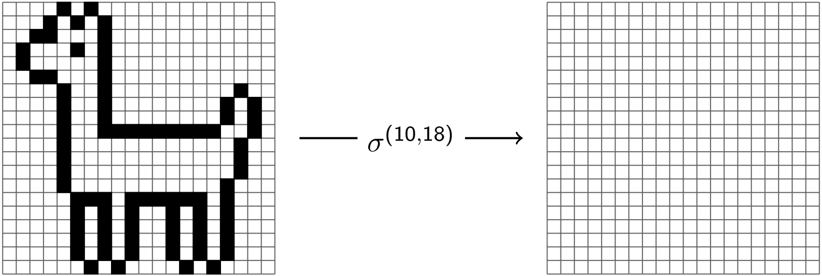



Figure: A random configuration $x \in \{\blacksquare, \square\}^{\mathbb{Z}^2/20\mathbb{Z}^2}$ and its image by $\sigma^{(10,18)}$.

# Example

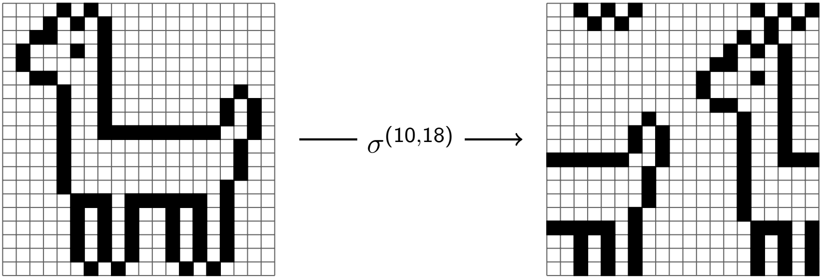



Figure: A random configuration $x \in \{\blacksquare, \square\}^{\mathbb{Z}^2/20\mathbb{Z}^2}$ and its image by $\sigma^{(10,18)}$.

# Context

**Definition: subshift**

$X \subset \mathcal{A}^G$ is a *subshift* or a *shift space* if and only if it is invariant under the action of $\sigma$ and closed for the product topology on $\mathcal{A}^G$.

# Context

**Definition: subshift**

$X \subset \mathcal{A}^G$ is a *subshift* or a *shift space* if and only if it is invariant under the action of $\sigma$ and closed for the product topology on $\mathcal{A}^G$.

Luckily, these objects can also be described in a combinatorial way.

# Context

**Definition: subshift**

$X \subset \mathcal{A}^G$ is a *subshift* or a *shift space* if and only if it is invariant under the action of $\sigma$ and closed for the product topology on $\mathcal{A}^G$.

Luckily, these objects can also be described in a combinatorial way.

**Proposition**

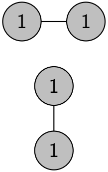
A subshift is a set of configurations avoiding patterns from a list $\mathcal{F}$.

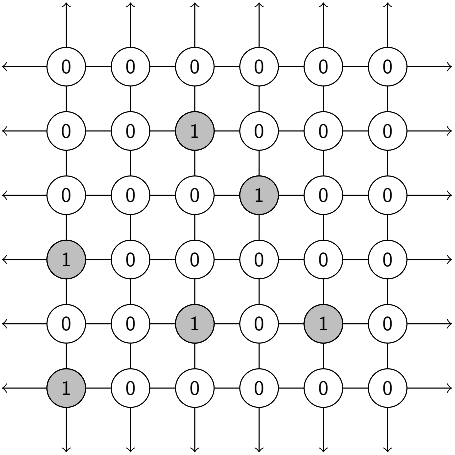
$$p \in \mathcal{A}^S, \qquad [p] = \{x \in \mathcal{A}^G \mid x|_S = p\}$$

$$X = X_{\mathcal{F}} = \mathcal{A}^G \setminus \bigcup_{g \in G, p \in \mathcal{F}} \sigma^g([p])$$

# Example in $\mathbb{Z}^2$: Hard square shift

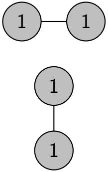
The set of assignments of $\mathbb{Z}^2$ to $\{0, 1\}$ such that there are no two adjacent ones.

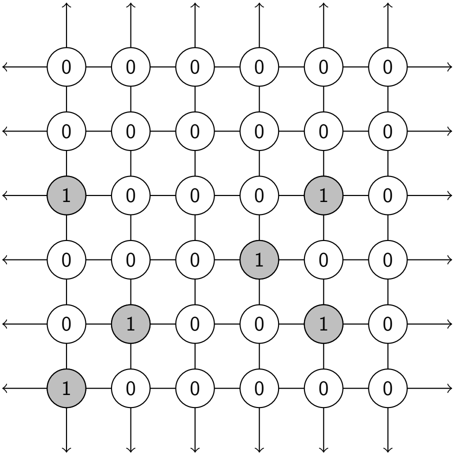
$\mathcal{F} =$ 

# Example in $\mathbb{Z}^2$: Hard square shift

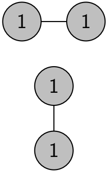
The set of assignments of $\mathbb{Z}^2$ to $\{0, 1\}$ such that there are no two adjacent ones.

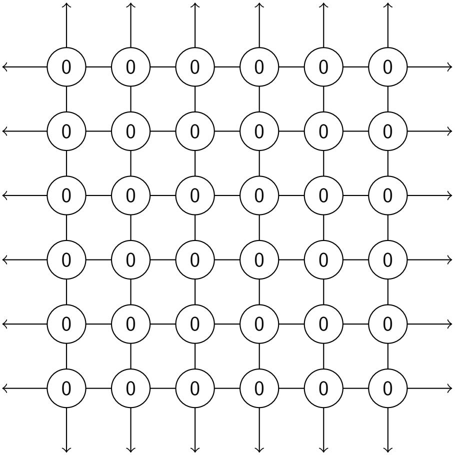
$\mathcal{F} =$ 

# Example in $\mathbb{Z}^2$: Hard square shift

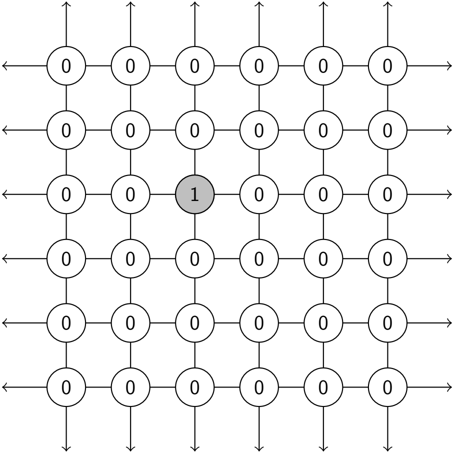
The set of assignments of $\mathbb{Z}^2$ to $\{0, 1\}$ such that there are no two adjacent ones.

$\mathcal{F} =$ 

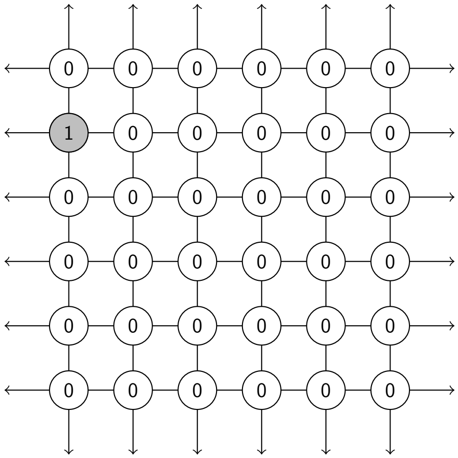
# Example: one-or-less subshift

$$X_{\leq 1} := \{x \in \{0,1\}^G \mid 0 \notin \{x_u, x_v\} \implies u = v\}.$$



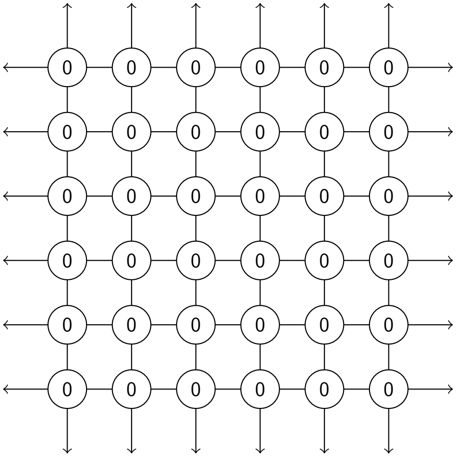
# Example: one-or-less subshift

$$X_{\leq 1} := \{x \in \{0,1\}^G \mid 0 \notin \{x_u, x_v\} \implies u = v\}.$$

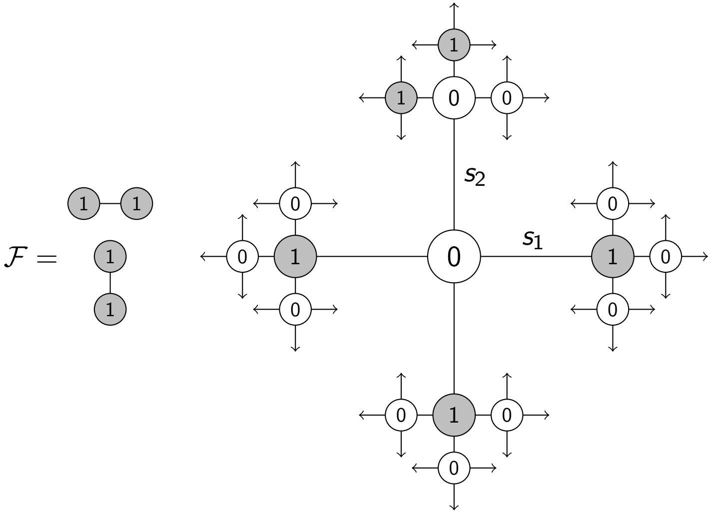


# Example: one-or-less subshift

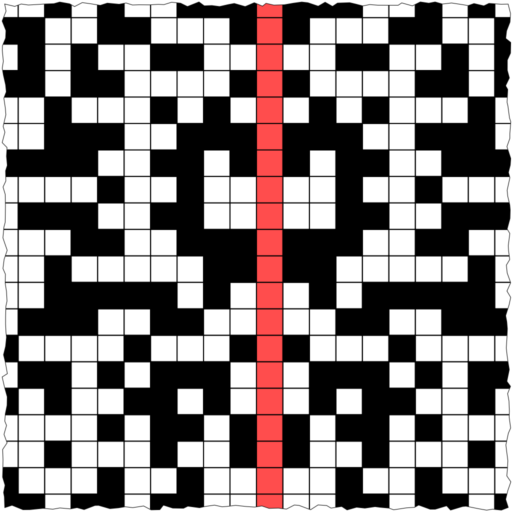
$$X_{\leq 1} := \{x \in \{0,1\}^G \mid 0 \notin \{x_u, x_v\} \implies u = v\}.$$



# Example: Hard square in $F_2$.



# Example: Mirror shift on $\mathbb{Z}^2$



# Definitions

## Classes of subshifts

A subshift $X \subset \mathcal{A}^G$ is called:

- a *subshift of finite type (SFT)* if $X = X_{\mathcal{F}}$ for some finite $\mathcal{F}$.
  **Ex: Hard square shift.**

# Definitions

## Classes of subshifts

A subshift $X \subset \mathcal{A}^G$ is called:

- a *subshift of finite type (SFT)* if $X = X_{\mathcal{F}}$ for some finite $\mathcal{F}$. **Ex: Hard square shift.**
- a *sofic subshift* if $X$ is the image of an SFT by a topological factor (a local recoding). **Ex: One-or-less in $\mathbb{Z}^2$.**

# Definitions

## Classes of subshifts

A subshift $X \subset \mathcal{A}^G$ is called:

- a *subshift of finite type (SFT)* if $X = X_{\mathcal{F}}$ for some finite $\mathcal{F}$. **Ex: Hard square shift.**
- a *sofic subshift* if $X$ is the image of an SFT by a topological factor (a local recoding). **Ex: One-or-less in $\mathbb{Z}^2$.**
- an *effectively closed subshift* if $X$ can be defined by a recursively enumerable coding of a set of forbidden patterns. **Ex: Mirror shift in $\mathbb{Z}^2$.**

# Why computability and dynamics?

**Dynamical objects**

$\triangleright$ Expansivity, periodic points, aperiodicity, invariant measures, entropy, recurrence, mixing, minimality, etc.

**Computational objects**

$\triangleright$ Finite information, description by Turing machines, computability of invariants, domino problem, simulation, etc.

# Why computability and dynamics?

### Dynamical objects

$\triangleright$ Expansivity, periodic points, aperiodicity, invariant measures, entropy, recurrence, mixing, minimality, etc.

### Computational objects

$\triangleright$ Finite information, description by Turing machines, computability of invariants, domino problem, simulation, etc.

The purpose of my thesis is to study the **interplay between computability and dynamics in shift spaces in groups**.

# Overview

**Some topics I worked on:**

- Aperiodic subshifts (and SFTs) on groups.
- Realization of subshifts with uniform densities.
- Effectively closed subshifts, subactions and simulation theorems.
- Notions of computability for subshifts on groups.
- Computability in automorphism groups and the topological full group of a shift space.

# Overview

**Some topics I worked on:**

- (1) Aperiodic subshifts (and SFTs) on countable groups.
- Realization of subshifts with uniform densities.
- **(1) Effectively closed subshifts, subactions and simulation theorems**.
- Notions of computability for subshifts on groups.
- **(2) Computability in automorphism groups and the topological full group of a shift space**.

Subactions, effectiveness, simulation and aperiodicity.

# Expansivity

**Definition**

An action $T : G \curvearrowright X$ is expansive if it separates distinct points.

$$\text{There is } C > 0, x \neq y \implies \exists g \in G, d(T^g(x), T^g(y)) > C$$

## Expansivity

### Definition

An action $T : G \curvearrowright X$ is expansive if it separates distinct points.

$$\text{There is } C > 0, x \neq y \implies \exists g \in G, d(T^g(x), T^g(y)) > C$$

### Odometers

Odometers are not expansive (the action is an isometry).

# Expansivity

## Definition

An action $T : G \curvearrowright X$ is expansive if it separates distinct points.

$$\text{There is } C > 0, x \neq y \implies \exists g \in G, d(T^g(x), T^g(y)) > C$$

## Odometers

Odometers are not expansive (the action is an isometry).

## Shift spaces

Every shift space is expansive
If $x \neq y$, there is $g \in G$ such that $x_g \neq y_g$. We have

$$\sigma^{g^{-1}}(x)|_{1_G} \neq \sigma^{g^{-1}}(y)|_{1_G}.$$

# What about the subactions of these classes?

Let $X \subset \mathcal{A}^G$ be a subshift and $H \leq_{\text{f.g.}} G$.

- ▷ What can we say about the system $(X, \sigma|_H)$?
- ▷ Same question when $X$ is an SFT, sofic or effectively closed.

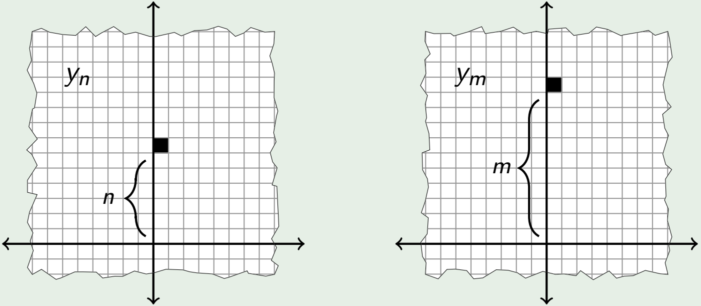
# What about the subactions of these classes?

Let $X \subset \mathcal{A}^G$ be a subshift and $H \leq_{\text{f.g.}} G$.

▷ What can we say about the system $(X, \sigma|_H)$?

▷ Same question when $X$ is an SFT, sofic or effectively closed.

**Remark: Subshifts are expansive, subactions not necessarily**




$$\inf_{n \neq m} d(\mathrm{Orb}_{\sigma|_{\mathbb{Z}}}(y_n), \mathrm{Orb}_{\sigma|_{\mathbb{Z}}}(y_m)) = 0$$

# Question 1: which are the subactions?

**Effectively closed action**

$T : G \curvearrowright X \subset \{0,1\}^{\mathbb{N}}$ is *effectively closed* if it can be described by Turing machines:

# Question 1: which are the subactions?

## Effectively closed action

$T : G \curvearrowright X \subset \{0,1\}^{\mathbb{N}}$ is *effectively closed* if it can be described by Turing machines:

- $X = \{0,1\}^{\mathbb{N}} \setminus \bigcup_{w \in L}[w]$, $L$ is recursively enumerable.
- There is a Turing machine which on entry $s \in G$ and $u \in \{0,1\}^*$ enumerates the complement of $T^s([u])$.

# Question 1: which are the subactions?

## Effectively closed action

$T : G \curvearrowright X \subset \{0,1\}^{\mathbb{N}}$ is *effectively closed* if it can be described by Turing machines:

- $X = \{0,1\}^{\mathbb{N}} \setminus \bigcup_{w \in L}[w]$, $L$ is recursively enumerable.
- There is a Turing machine which on entry $s \in G$ and $u \in \{0,1\}^*$ enumerates the complement of $T^s([u])$.

Every subaction of an effectively closed subshift (also sofic/SFT) is effectively closed.

# Question 2: can we realize e.c.d.s ?

**Question**

Given an effectively closed dynamical system. Can we realize it as a subaction of an SFT/sofic subshift?

# Question 2: can we realize e.c.d.s ?

**Question**

Given an effectively closed dynamical system. Can we realize it as a subaction of an SFT/sofic subshift?

Before answering that question, let us motivate this kind of results:

# The philosophy behind it

## Finitely presented group

A group $G$ is finitely presented if it can be described as $G = \langle S | R \rangle$ where both $S$ and $R \subset (S \cup S^{-1})^*$ are finite.

$$\mathbb{Z}^2 \cong \langle a, b \mid aba^{-1}b^{-1} \rangle.$$

$$BS(1,2) \cong \langle a, b \mid aba^{-2}b^{-1} \rangle.$$

# The philosophy behind it

## Finitely presented group

A group $G$ is finitely presented if it can be described as $G = \langle S|R\rangle$ where both $S$ and $R \subset (S \cup S^{-1})^*$ are finite.

$$\mathbb{Z}^2 \cong \langle a, b \mid aba^{-1}b^{-1}\rangle.$$
$$BS(1,2) \cong \langle a, b \mid aba^{-2}b^{-1}\rangle.$$

## Recursively presented group

A group $G$ is recursively presented if it can be described as $G = \langle S|R\rangle$ where $S \subset \mathbb{N}$ and $R \subset (S \cup S^{-1})^*$ are recursive sets.

$$L \cong \langle a, t \mid (at^n at^{-n})^2, n \in \mathbb{N}\rangle.$$
$$\bigoplus_{i\in\mathbb{N}}(\mathbb{Z}/2\mathbb{Z}) \cong \langle \{a_n\}_{n\in\mathbb{N}} \mid \{a_n^2\}_{n\in\mathbb{N}} \cup \{a_j a_k a_j^{-1} a_k^{-1}\}_{j,k\in\mathbb{N}}\rangle.$$

# The philosophy behind it

**Theorem (Higman 1961)**

*For every recursively presented group $H$ there exists a finitely presented group $G$ such that $H$ embeds into $G$.*

## The philosophy behind it

**Theorem (Higman 1961)**

*For every recursively presented group $H$ there exists a finitely presented group $G$ such that $H$ embeds into $G$.*

"A complicated object is realized inside another object which admits a much simpler presentation."

# The philosophy behind it

**Theorem (Higman 1961)**

*For every recursively presented group $H$ there exists a finitely presented group $G$ such that $H$ embeds into $G$.*

"A complicated object is realized inside another object which admits a much simpler presentation."

**word problem**: is it algorithmically decidable if a word over a set of generators of a group represents the identity?

**Theorem: (Novikov 1955, Boone 1958)**

There are finitely presented groups with undecidable word problem

# The philosophy behind it

**Theorem (Higman 1961)**

*For every recursively presented group $H$ there exists a finitely presented group $G$ such that $H$ embeds into $G$.*

"A complicated object is realized inside another object which admits a much simpler presentation."

**word problem**: is it algorithmically decidable if a word over a set of generators of a group represents the identity?

**Theorem: (Novikov 1955, Boone 1958)**

There are finitely presented groups with undecidable word problem

Just apply Higman's theorem to
$G = \langle a, b, c, d \mid b^{-n}ab^n = c^{-n}dc^n, n \in \mathtt{HALT}\rangle$... done!

# Two recent simulation results

**Theorem (Hochman 2009)**

For every effectively closed action $T : \mathbb{Z}^d \curvearrowright X \subset \{0,1\}^{\mathbb{N}}$ there exists a $\mathbb{Z}^{d+2}$-SFT $\hat{X}$ such that one of its $\mathbb{Z}^d$-subactions is an extension of $T$.

$$\begin{tikzcd}
\mathbb{Z}^{d+2} & (\hat{X}, \sigma) \arrow[d, "\text{subaction}"'] & \\
\mathbb{Z}^{d} & (\hat{X}, \sigma|_{\mathbb{Z}^d}) \arrow[r, two heads, "\text{factor}"'] & (X, T)
\end{tikzcd}$$

Moreover, the factor is small.

# Two recent simulation results

One can do better in the expansive case:

**Theorem (Aubrun-Sablik 2010, Durand-Romaschenko-Shen 2010)**

Every effectively closed $\mathbb{Z}^d$-subshift is the subaction (projective subaction) of a $\mathbb{Z}^{d+1}$-sofic subshift.

$$\begin{tikzcd}
\color{red}{\mathbb{Z}^{d+1}} & (\hat{X},\sigma) \arrow[rr, two heads, "\text{symb factor}"'] & & (\hat{Y},\sigma) \arrow[d, "\text{subaction}"'] \\
\mathbb{Z}^d & & & (X,T)
\end{tikzcd}$$

# An application: strongly aperiodic subshifts

**Definition (Strongly aperiodic subshift)**

A subshift $X \subset \mathcal{A}^G$ is *strongly aperiodic* if the shift action is free

$$\forall x \in X, \forall g \in G, \ \sigma^g(x) = x \Rightarrow g = 1_G.$$

# An application: strongly aperiodic subshifts

**Definition (Strongly aperiodic subshift)**

A subshift $X \subset \mathcal{A}^G$ is *strongly aperiodic* if the shift action is free

$$\forall x \in X, \forall g \in G,\ \sigma^g(x) = x \Rightarrow g = 1_G.$$

**Proposition**

Every 1D non-empty SFT contains a periodic configuration.

# An application: strongly aperiodic subshifts

**Definition (Strongly aperiodic subshift)**

A subshift $X \subset \mathcal{A}^G$ is *strongly aperiodic* if the shift action is free

$$\forall x \in X, \forall g \in G, \ \sigma^g(x) = x \Rightarrow g = 1_G.$$

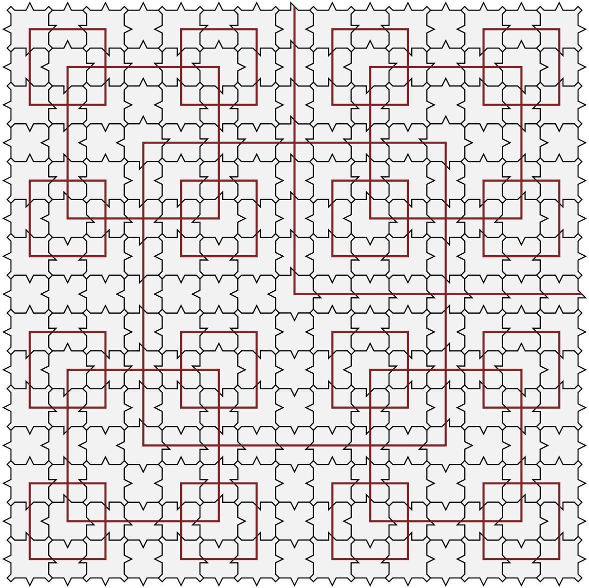
**Proposition**

Every 1D non-empty SFT contains a periodic configuration.

**Theorem (Berger 1966, Robinson 1971, Kari 1996, Jeandel-Rao 2015)**

There exist strongly aperiodic SFTs on $\mathbb{Z}^2$.

# Example of strongly aperiodic $\mathbb{Z}^2$-SFT: Robinson tileset

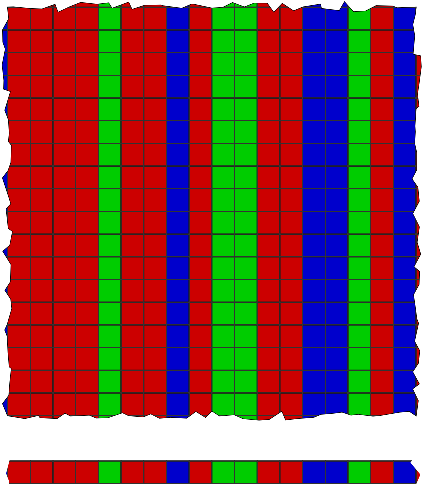


# Simulation in the case of subshifts

$\mathbb{Z}^2$-sofic

**p. subaction**

$\mathbb{Z}$-effective



$\in Y$

$\in X$

# So... why is simulation important?

It is complicated to come up with $\mathbb{Z}^2$-SFTs which are strongly aperiodic, however, finding a $\mathbb{Z}$-effectively closed subshift which is aperiodic is easy.

# So... why is simulation important?

It is complicated to come up with $\mathbb{Z}^2$-SFTs which are strongly aperiodic, however, finding a $\mathbb{Z}$-effectively closed subshift which is aperiodic is easy.

**Example**

Let $x$ be a fixed point of the Thue-Morse substitution.

$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \dots$$

Then $X = \overline{\mathrm{Orb}_{\sigma}(x)}$ is strongly aperiodic and effectively closed.

# So... why is simulation important?

It is complicated to come up with $\mathbb{Z}^2$-SFTs which are strongly aperiodic, however, finding a $\mathbb{Z}$-effectively closed subshift which is aperiodic is easy.

**Example**

Let $x$ be a fixed point of the Thue-Morse substitution.

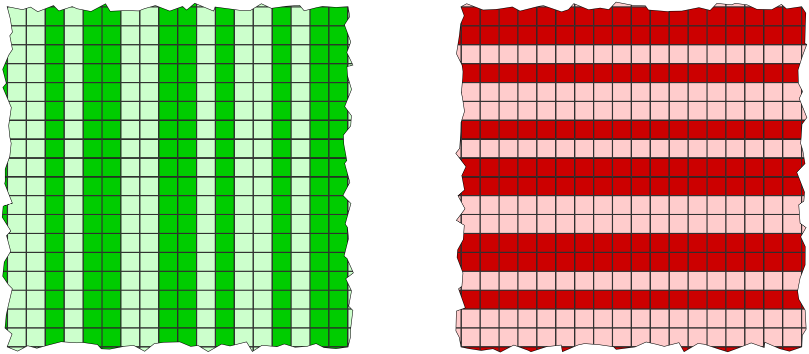
$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \dots$$

Then $X = \overline{\mathrm{Orb}_\sigma(x)}$ is strongly aperiodic and effectively closed.

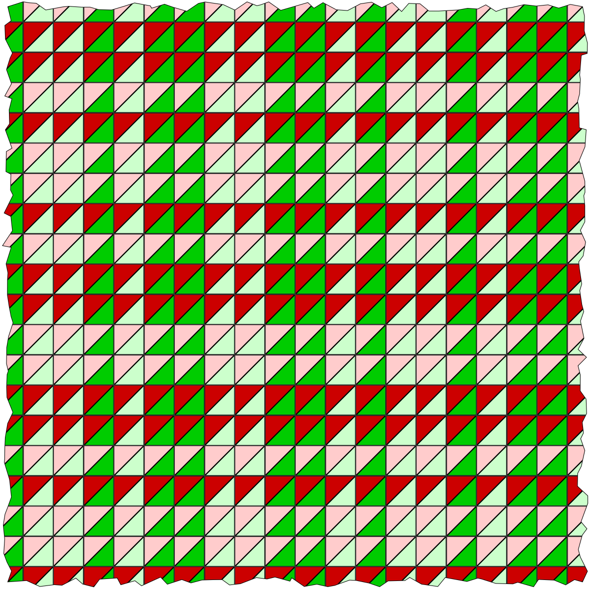
**Example**

A Sturmian subshift defined by an irrational computable slope $\alpha$.

# So... why is simulation important?



# So... why is simulation important?



# So... why is simulation important?

**Examples**

- Easy construction of strongly aperiodic $\mathbb{Z}^2$-SFTs

# So... why is simulation important?

**Examples**

- Easy construction of strongly aperiodic $\mathbb{Z}^2$-SFTs
- $\mathbb{Z}^2$-SFTs with no computable configurations (Original result by Hanf-Myers 1974)

# So... why is simulation important?

**Examples**

- Easy construction of strongly aperiodic $\mathbb{Z}^2$-SFTs
- $\mathbb{Z}^2$-SFTs with no computable configurations (Original result by Hanf-Myers 1974)
- Classifying the entropies of $\mathbb{Z}^2$-SFTs (Original result by Hochman-Meyerovitch 2010)

What if we were able to do this in general groups?

# Two new results in general groups

Let $T : G \curvearrowright X \subset \{0,1\}^{\mathbb{N}}$ be an effectively closed action of a finitely generated group.

**Theorem (B-Sablik, 2016)**

*For any semidirect product $\mathbb{Z}^2 \rtimes G$ there exists a $\mathbb{Z}^2 \rtimes G$-SFT such that its $G$-subaction is an extension of $T$.*

# Two new results in general groups

Let $T : G \curvearrowright X \subset \{0,1\}^{\mathbb{N}}$ be an effectively closed action of a finitely generated group.

**Theorem (B-Sablik, 2016)**

*For any semidirect product $\mathbb{Z}^2 \rtimes G$ there exists a $\mathbb{Z}^2 \rtimes G$-SFT such that its $G$-subaction is an extension of $T$.*

**Theorem (B, 2017)**

*For any pair of infinite and finitely generated groups $H_1, H_2$ there exists a $(G \times H_1 \times H_2)$-SFT such that its $G$-subaction is an extension of $T$.*

# Strongly aperiodic SFTs in these groups.

We want to go ahead and use the simulation theorems to produce strongly aperiodic SFTs, but...

# Strongly aperiodic SFTs in these groups.

We want to go ahead and use the simulation theorems to produce strongly aperiodic SFTs, but...

**Question**

- For which groups can I find an effectively closed strongly aperiodic subshift?

# Strongly aperiodic SFTs in these groups.

We want to go ahead and use the simulation theorems to produce strongly aperiodic SFTs, but...

**Question**

- For which groups can I find an effectively closed strongly aperiodic subshift?

In fact...

**Question by Glasner and Uspenskij 2009**

- Is there a countable group which admits no aperiodic subshifts over the alphabet $\mathcal{A} = \{0, 1\}$?

# Strongly aperiodic subshifts?

**Question by Glasner and Uspenskij 2009**

Is there a countable group which admits no aperiodic subshifts over the alphabet $\mathcal{A} = \{0,1\}$?

# Strongly aperiodic subshifts?

**Question by Glasner and Uspenskij 2009**

Is there a countable group which admits no aperiodic subshifts over the alphabet $\mathcal{A} = \{0, 1\}$?

**Theorem (Gao-Jackson-Seward, 2009)**

*No! All do.*

And the proof is a little bit technical.

# Strongly aperiodic subshifts?

**Question by Glasner and Uspenskij 2009**

Is there a countable group which admits no aperiodic subshifts over the alphabet $\mathcal{A} = \{0, 1\}$?

**Theorem (Gao-Jackson-Seward, 2009)**

*No! All do.*

And the proof is a little bit technical.

**Theorem (Aubrun-B-Thomassé, 2015)**

*No! All do.*

But the proof is much shorter, moreover, the construction is effective for groups with decidable word problem. It is based on a probabilistic argument using Lovász local lemma.

# Strongly aperiodic subshifts?

Moreover, a recent result by Jeandel states:

**Theorem (Jeandel, 2015)**

*Let $G$ be a recursively presented group. If $G$ admits an effectively closed strongly aperiodic subshift then its word problem is decidable.*

# Strongly aperiodic subshifts?

Moreover, a recent result by Jeandel states:

**Theorem (Jeandel, 2015)**

*Let $G$ be a recursively presented group. If $G$ admits an effectively closed strongly aperiodic subshift then its word problem is decidable.*

With our result we can write:

**Theorem (Aubrun-B-Thomassé, 2015)**

*Let $G$ be a recursively presented group. $G$ admits an effectively closed strongly aperiodic subshift if and only if its word problem is decidable.*

# Strongly aperiodic SFTs!

**Theorem (B-Sablik, 2016)**

*If $G$ is finitely generated, has decidable word problem and $d > 1$. Then any group of the form $\mathbb{Z}^d \rtimes_{\varphi} G$ admits a SA SFT.*

**Ex: The discrete Heisenberg group.**

# Strongly aperiodic SFTs!

**Theorem (B-Sablik, 2016)**

*If $G$ is finitely generated, has decidable word problem and $d > 1$. Then any group of the form $\mathbb{Z}^d \rtimes_\varphi G$ admits a SA SFT.*

**Ex: The discrete Heisenberg group.**

**Theorem (B, 2017)**

*If $G_i$ are at least three infinite and finitely generated groups with decidable word problem. Then $G_1 \times \cdots \times G_n$ admits a SA SFT.*

**Ex: (not an obvious corollary) The Grigorchuk group.**

# A negative result

Recall the subshift:

$$X_{\leq 1} = \{x \in \{0,1\}^G \mid 0 \notin \{x_g, x_h\} \implies g = h\}$$

configurations $x \in X_{\leq 1}$ have at most one occurrence of 1.

# A negative result

Recall the subshift:

$$X_{\leq 1} = \{x \in \{0,1\}^G \mid 0 \notin \{x_g, x_h\} \implies g = h\}$$

configurations $x \in X_{\leq 1}$ have at most one occurrence of 1.

**Theorem (Aubrun-B-Sablik 2015)**

*If $G$ is finitely generated and recursively presented, then $X_{\leq 1}$ is effectively closed if and only if $G$ has decidable word problem.*

In particular, there are groups for which $X_{\leq 1}$ is not even conjugate to the subaction of an effectively closed subshift.

# Computability in group invariants of shift spaces

# Motivation

Given a subshift $(X, \sigma)$, its automorphism group is given by

$$\mathrm{Aut}(X) = \{\phi : X \to X \text{ homeomorpism}, [\sigma, \phi] = \mathrm{id}\}$$

# Motivation

Given a subshift $(X, \sigma)$, its automorphism group is given by

$$\mathrm{Aut}(X) = \{\phi : X \to X \text{ homeomorpism}, [\sigma, \phi] = \mathrm{id}\}$$

▷ $\mathrm{Aut}(\mathcal{A}^G)$ is the group of reversible cellular automata.

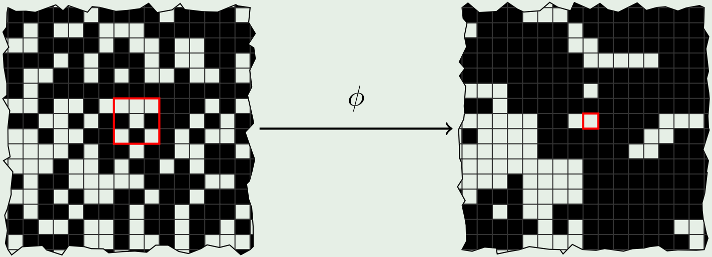
# Motivation

Given a subshift $(X, \sigma)$, its automorphism group is given by

$$\mathrm{Aut}(X) = \{\phi : X \to X \text{ homeomorpism}, [\sigma, \phi] = \mathrm{id}\}$$

▷ $\mathrm{Aut}(\mathcal{A}^G)$ is the group of reversible cellular automata.

**Example and non-example.**



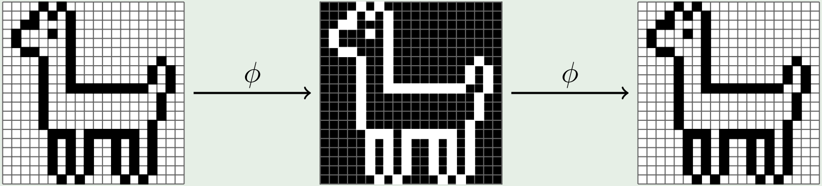
# Motivation

Given a subshift $(X, \sigma)$, its automorphism group is given by

$$\mathrm{Aut}(X) = \{\phi : X \to X \text{ homeomorpism}, [\sigma, \phi] = \mathrm{id}\}$$

▷ $\mathrm{Aut}(\mathcal{A}^G)$ is the group of reversible cellular automata.

**Example and non-example.**



# Motivation

The topological fullgroup of a dynamical system $(X, T)$ where $T : G \curvearrowright X$ is

$$[[T]] = \{\phi \in \mathrm{Homeo}(X) \mid \exists s : X \to G \text{ continuous}, \phi(x) = T^{s(x)}(x)\}.$$

# Motivation

The topological fullgroup of a dynamical system $(X, T)$ where $T : G \curvearrowright X$ is

$$[[T]] = \{\phi \in \mathrm{Homeo}(X) \mid \exists s : X \to G \text{ continuous}, \phi(x) = T^{s(x)}(x)\}.$$

This group is the restriction of a larger group introduced as an invariant of orbit equivalence.

# Motivation

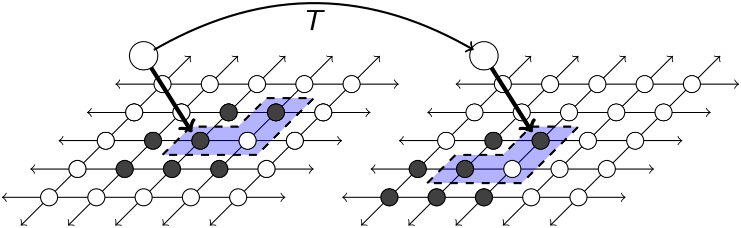
The topological fullgroup of a dynamical system $(X, T)$ where $T : G \curvearrowright X$ is

$$[[T]] = \{\phi \in \text{Homeo}(X) \mid \exists s : X \to G \text{ continuous}, \phi(x) = T^{s(x)}(x)\}.$$

This group is the restriction of a larger group introduced as an invariant of orbit equivalence.

In the case of a subshift it can be interpreted as a group of abstract Turing machines which are globally reversible and do not change the tape.

# Motivation




$s(\bullet\overset{\bullet}{\circ}) = (1, 1)$

# Motivation

It might be a good idea to understand these groups from a computability point of view:

**Definition (three problems)**

Let $G = \langle S \mid R \rangle$ be a finitely generated group.

- **Word problem:** Given $w \in S^*$, is $w$ the identity of the group?
- **Torsion problem:** Given $w \in S^*$, is there $n \in \mathbb{N}$ such that $w^n$ is the identity of the group?
- **Finiteness problem:** Given $w_1, \ldots, w_k \in S^*$, is the group generated by them finite?

# Motivation

It might be a good idea to understand these groups from a computability point of view:

**Definition (three problems)**

Let $G = \langle S \mid R \rangle$ be a finitely generated group.

- **Word problem:** Given $w \in S^*$, is $w$ the identity of the group?
- **Torsion problem:** Given $w \in S^*$, is there $n \in \mathbb{N}$ such that $w^n$ is the identity of the group?
- **Finiteness problem:** Given $w_1, \ldots, w_k \in S^*$, is the group generated by them finite?

The real problem is that $[[\sigma]]$ and $\mathrm{Aut}(X)$ in general are not finitely generated...

# Two weird examples

▷ $K \subset \mathbb{N}$ be r.e. but undecidable.

**Example 1**: decidability depends on the presentation.

$$
\begin{aligned}
G &\cong \langle \{a_n\}_{n\in\mathbb{N}} \mid \{[a_n, a_m]\}_{n,m\in\mathbb{N}} \cup \{(a_k)^2\}_{k\in K} \rangle \\
&\cong \langle \{b_n\}_{n\in\mathbb{N}} \mid \{[b_n, b_m]\}_{n,m\in\mathbb{N}} \cup \{(b_\ell)^2\}_{\ell \text{ is even }} \rangle.
\end{aligned}
$$

# Two weird examples

▷ $K \subset \mathbb{N}$ be r.e. but undecidable.

**Example 1**: decidability depends on the presentation.

$$\begin{aligned} G &\cong \langle \{a_n\}_{n\in\mathbb{N}} \mid \{[a_n, a_m]\}_{n,m\in\mathbb{N}} \cup \{(a_k)^2\}_{k\in K} \rangle \\ &\cong \langle \{b_n\}_{n\in\mathbb{N}} \mid \{[b_n, b_m]\}_{n,m\in\mathbb{N}} \cup \{(b_\ell)^2\}_{\ell \text{ is even}} \rangle. \end{aligned}$$

**Example 2**: No presentation with decidable WP, all f.g. subgroups have decidable WP.

▷ $\{p_n\}_{n\in\mathbb{N}} =$ primes $1 \mod 4$.

▷ $\{q_n\}_{n\in\mathbb{N}} =$ primes $3 \mod 4$.

$$G = \langle \{a_n\}_{n\in\mathbb{N}} \mid \{[a_n, a_m], (a_n)^{p_n q_n}\}_{n,m\in\mathbb{N}} \cup \{(a_k)^{p_k}\}_{k\in K} \rangle.$$

# Some results

**Theorem (B-Kari-Salo, 2016)**

*For any finite alphabet $A$ with at least two symbols, $\mathrm{Aut}(A^{\mathbb{Z}})$ contains a finitely generated subgroup with undecidable torsion problem.*

# Some results

**Theorem (B-Kari-Salo, 2016)**

*For any finite alphabet $A$ with at least two symbols, $\mathrm{Aut}(A^{\mathbb{Z}})$ contains a finitely generated subgroup with undecidable torsion problem.*

**Theorem (B-Kari-Salo, 2016)**

*The same is true for the topological full group of $(A^{\mathbb{Z}^d}, \sigma)$ if and only if $d \geq 2$.*

# Perspectives

## Short and middle term

- Simulation theorem $G \times \mathbb{Z}$ or $G \times H$ for expansive actions.
- Strongly aperiodic subshifts in products $G_1 \times G_2$? (with $\mathtt{WP}(G_i)$ decidable)
- Minimal effectively closed strongly aperiodic subshifts?
- Other applications: entropies of SFTs, realize mixing conditions, etc.
- Study unavoidable sets of patterns.
- Dynamical proof of Higman's theorem.

# Perspectives (wishes)

## Long term (related to this talk)

- Characterize the groups which admit strongly aperiodic SFTs.
- Characterize the entropies of SFTs on f.g. groups (amenable... or maybe sofic).

## Long term (unrelated to this talk)

- The equal entropy SFT cover problem.
- Domino problem for f.g. groups
- Better understand conjugacy of automorphism groups: the $\mathrm{Aut}(\{0,1\}^{\mathbb{Z}})$ vs $\mathrm{Aut}(\{0,1,2\}^{\mathbb{Z}})$ problem.

Thank you for your attention!

Merci pour votre attention !

¡Gracias por su atención!